FORMULA

Differentiability

1. $\frac{d(x^n)}{dx} = nx^{n-1}$

2. $\frac{d\{cf(x)\}}{dx} = c\frac{d(f(x))}{dx}$

3. $\frac{d(a^x)}{dx} = a^x \log_e a$

4. $\frac{d(e^x)}{dx} = e^x$

5. $\frac{d(\log_e x)}{dx} = \frac{1}{x}$

6. $\frac{d(u+v+w)}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$

7. $\frac{d(u-v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

8. $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

9. $\frac{d(uvw)}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

10. $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

11. $\frac{d(\sin x)}{dx} = \cos x$

12. $\frac{d(\cos x)}{dx} = -\sin x$

13. $\frac{d(\tan x)}{dx} = \sec^2 x$

14. $\frac{d(\sec x)}{dx} = \sec x \tan x$

15. $\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

16. $\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$

17. $\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$

18. $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

19. $\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$

20. $\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$

21. $\frac{d(\cot^{-1} x)}{dx} = \frac{-1}{1+x^2}$

22. $\frac{d(\sec^{-1} x)}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

23. $\frac{d(\operatorname{cosec}^{-1} x)}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$

24. $\frac{d(\log_a x)}{dx} = \frac{1}{x}\log_a e = \frac{1}{x\log_e a}$

25.) Differentiation for Implicit functions :-

$$\frac{dy}{dx} = -\left(\frac{\partial f/\partial x}{\partial f/\partial y}\right)$$

26.) Right Hand derivative :-

$$f'(c^+) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

Left Hand derivative :-

$$f'(c^-) = \lim_{h\to 0} \frac{f(c-h) - f(c)}{-h}$$

27.) $f(x) = \begin{cases} x^n \sin(1/x) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

if $n \le 1$ then $f(x)$ is continuous But not differentiable
if $n > 1$ then $f(x)$ is continuous and differentiastie.

28.) Continuous :-

$$\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+h) = f(a) = \text{finite}$$

29.) Differentiation of Determinant :-

$$F(x) = \begin{vmatrix} f(x) & g(x) \\ h(x) & w(x) \end{vmatrix}$$

then $F'(x) = \begin{vmatrix} f'(x) & g'(x) \\ h(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ h'(x) & w'(x) \end{vmatrix}$

30.) $g(f(x)) = f(g(x)) = x$

Hyperbolic :-

31.) $\frac{d(\sinh^{-1}x)}{dx} = \frac{1}{\sqrt{x^2+1}}$

32.) $\frac{d(\cosh^{-1}x)}{dx} = \frac{1}{\sqrt{x^2-1}}$

33.) $\frac{d(\tanh^{-1}x)}{dx} = \frac{1}{1-x^2}$

34.) $\frac{d(\coth^{-1}x)}{dx} = \frac{-1}{1-x^2}$

35.) $\frac{d(\text{sech}^{-1}x)}{dx} = \frac{-1}{x\sqrt{1-x^2}}$

36.) $\frac{d(\text{cosech}^{-1}x)}{dx} = \frac{-1}{\sqrt{1+x^2}}$

Integration

1. $\int a \cdot dx = ax + c$

3. $\int e^x \cdot dx = e^x + c$

2. $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c$

4. $\int \frac{1}{x} \cdot dx = \log_e x + c$

5. $\int \cos x \cdot dx = \sin x + c$

6. $\int \sin x \cdot dx = -\cos x + c$

7. $\int \sec^2 x \cdot dx = \tan x + c$

8. $\int \sec x \tan x \cdot dx = \sec x + c$

9. $\int \operatorname{cosec}^2 x \cdot dx = -\cot x + c$

10. $\int \operatorname{cosec} x \cot x \cdot dx = -\operatorname{cosec} x + c$

11. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

12. $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + c$

13. $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$

14. $\int \frac{dx}{1+x^2} = -\cot^{-1} x + c$

15. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$

16. $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + c$

17. $\int a^x dx = \frac{a^x}{\log_e a}$

18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

19. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

20. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

21. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + c$

22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + c$

23. $\int \frac{1}{a^2-x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

24. $\int \frac{1}{x^2-a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

25. $\int |x| \, dx = \frac{x|x|}{2} + c$

26. $\int \sqrt{x^2+a^2} \cdot dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \log\left|x + \sqrt{x^2+a^2}\right| + c$

27. $\int \sqrt{x^2-a^2} \cdot dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \cancel{\sin^{-1}\left(\frac{x}{a}\right)} \log\left|x + \sqrt{x^2-a^2}\right| + c$

27.) $\int \sqrt{a^2-x^2}\cdot dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

28.) $\int \tan x \cdot dx = \log|\sec x| + c$

29.) $\int \cot x \cdot dx = \log|\sin x| + c$

30.) $\int \operatorname{cosec} x \cdot dx = \log|\operatorname{cosec} x - \cot x| + c = \log\left|\tan\frac{x}{2}\right| + c$

31.) $\int \sec x \cdot dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$

32.) $\int \frac{f'(x)}{f(x)}\cdot dx = \log|f(x)| + c$

33.) $\int (f(x))^n \cdot f'(x)\cdot dx = \frac{f(x)^{n+1}}{n+1} + c$

34.) (A) $2\sin x\cos y = \sin(x+y) + \sin(x-y)$

(B) $2\cos x\sin y = \sin(x+y) - \sin(x-y)$

(C) $2\cos x\cos y = \cos(x+y) + \cos(x-y)$

(D) $2\sin x\sin y = \cos(x-y) - \cos(x+y)$

35.) Partial fraction in Integration

(Expression)	(Partial fraction)
(A) $\frac{Px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
(B) $\frac{Px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
(C) $\frac{Px+q}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
(D) $\frac{Px^2+q}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
(E) $\frac{Px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

36| Standrd Type of integration :

Type-① $\int \frac{1}{ax^2+bx+c}\cdot dx$, $\int \frac{1}{\sqrt{ax^2+bx+c}}\cdot dx$, $\int \sqrt{ax^2+bx+c}\cdot dx$

ax^2+bx+c को पूर्ण वर्ग में परिवर्तित कर हल करें -

Type-② $\int \frac{px+q}{ax^2+bx+c}\cdot dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\cdot dx$, $\int (px+q)\sqrt{ax^2+bx+c}\cdot dx$

let $px+q = A\frac{d}{dx}(ax^2+bx+c) + B = A(2ax+b)+B$

then find A & B.

Type-③ $\int \frac{1}{a+b\sin^2x}\cdot dx$, $\int \frac{1}{a+b\cos^2x}\cdot dx$,

$\int \frac{1}{a\sin^2x + b\cos^2x + c\sin x\cos x + d}\cdot dx$ then divided by $\cos^2x$ in N^r & DN^r then

put $\tan x = t$

Type-④ $\int \frac{1}{a\sin x+b}\cdot dx$, $\int \frac{1}{a\cos x+b}\cdot dx$, $\int \frac{1}{a\sin x+b\cos x+c}\cdot dx$

Solve the integration then $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$,

$\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ then put $\tan x/2 = t$

Type-⑤ $\int \frac{a\sin x + b\cos x}{p\sin x + q\cos x}\cdot dx$

$a\sin x + b\cos x = A(p\sin x + q\cos x) + B(p\cos x - q\sin x)$

find A & B then

$I = Ax + B\log|p\sin x + q\cos x| + c$

POPU

Page No.:
Date:

Type-⑥ $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r} \cdot dx$

$a\sin x + b\cos x + c = A(p\sin x + q\cos x + r) + B(p\cos x - q\sin x) + c$

Type-⑦ $\int \frac{x^2 \pm 1}{x^4 + kx^2 + 1} \cdot dx$ divide by x^2 in N^r & DN^r

$\int \frac{1 \pm 1/x^2}{x^2 + \frac{1}{x^2} + k} \cdot dx$, $x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2 = (x - \frac{1}{x})^2 + 2$

37.| Integration By parts :—

ILATE

$\int \underset{\mathrm{I}}{f(x)} \cdot \underset{\mathrm{II}}{g(x)} \cdot dx = f(x)\int g(x) \cdot dx - \int \left\{ \frac{df(x)}{dx} \int g(x) \cdot dx \right\} dx$

(A) $\int e^x (f(x) + f'(x)) \cdot dx = e^x f(x) + c$

(B) $\int (x f'(x)) + f(x) \cdot dx = x f(x)$

38.| Special Integrals :—

⇒ $\frac{1}{L\sqrt{L}}$, $\frac{1}{Q\sqrt{L}}$, $\sqrt{L} = t$ (let)

⇒ $\frac{1}{L\sqrt{Q}}$ then let $L = \frac{1}{t}$ (आ square में $x = \tan u$)

⇒ $\frac{1}{Q\sqrt{Q}}$ Let $x = \frac{1}{t}$

39.| $\int \frac{\sin x \pm \cos x}{f(\sin 2x)} \cdot dx$

$\sin 2x = (\sin x + \cos x)^2 - 1 = 1 - (\sin x - \cos x)^2$

40.) Properties of definite Integration ⇒

$P_0 \Rightarrow \quad \int_a^b f(x)\cdot dx = \int_a^b f(t)\cdot dt$

$P_1 \Rightarrow \quad \int_a^b f(x)\cdot dx = -\int_b^a f(x)\cdot dx$

$P_2 \Rightarrow \quad \int_a^b f(x)\cdot dx = \int_a^c f(x)\cdot dx + \int_c^b f(x)\cdot dx$

$P_3 \Rightarrow \quad \int_a^b f(x)\cdot dx = \int_a^b f(a+b-x)\cdot dx$

$P_4 \Rightarrow \quad \int_0^a f(x)\cdot dx = \int_0^a f(a-x)\cdot dx$

$P_5 \Rightarrow \quad \int_0^{2a} f(x)\cdot dx = \int_0^a f(x)\cdot dx + \int_0^a f(2a-x)\cdot dx$

$P_6 \Rightarrow \quad \int_0^{2a} f(x)\cdot dx = 2\int_0^a f(x)\cdot dx \qquad \{f(2a-x) = f(x)\}$

$P_7 \Rightarrow \quad \int_{-a}^a f(x)\cdot dx = 2\int_0^a f(x)\cdot dx$ if funtion is even function $\{f(x) = f(-x)\}$

if function is odd function $f(x) = -f(-x)$
then $\int_{-a}^a f(x)\cdot dx = 0$

41.) x के निष्कासन का नियम :—

$$\int_a^b x\, f(x)\cdot dx = \frac{a+b}{2}\int_a^b f(x)\cdot dx$$

$$\{\because f(a+b-x) = f(x)\}$$

42.) ~~e^m to~~ $e^{m\log x} = x^m$

43.) योग की सीमा के रूप में निश्चित समाकलन

$$\int_a^b f(x)\cdot dx = \lim_{h\to 0} h\left[f(a+h) + f(a+2h) + \cdots + f(a+nh)\right]$$

$\boxed{b-a = nh}$, $n \to \infty$, $h \to 0$

$$\boxed{\int_a^b f(x).dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} hf(a+rh)}$$

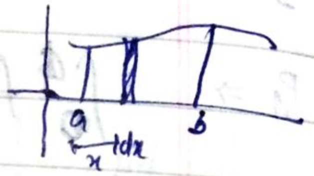
if $a=0, b=1$
$b-a=1$

$$\boxed{\lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\cdot\frac{1}{n} = \int_0^1 f(x).dx}$$

44. Area Bounded by curves:—

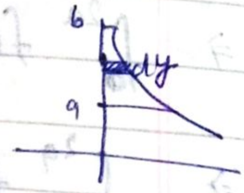
(i) Vertical Slicing:—
if $f(x) \geq 0$ x-axis, $x \in [a,b]$

$$\int_a^b y.dx = \int_a^b f(x).dx$$



(ii) Horizontal Slicing:—
if $f(y) \geq 0$, y-axis $y \in [a,b]$

$$\int_a^b x.dy = \int_a^b f(y).dy$$



45. $$\int_\alpha^\beta f(x).dx + \int_{f(\alpha)}^{f(\beta)} f^{-1}(x).dx = \beta f(\beta) - \alpha f(\alpha)$$

46. Rolle's theorem:—
(i) प्रांत संतत व अवकलनीय हो।
(ii) $f(a) = f(b)$ & $f'(c) = 0$ कर c का मान $c \in [a,b]$

47. Mean Value theorem:—
(i) प्रांत संतत व अवकलनीय हो।
(ii) $f(a) \neq f(b)$
(iii) $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ कर $c \in [a,b]$

Engineer Mathematics

Q.1 $z = \tan^{-1}(x/y)$ solve $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

Ans-
$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial x} = \frac{y^2}{y^2+x^2} \cdot \left(\frac{y \cdot 1 - x \cdot 0}{y^2}\right) = \frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y} = \frac{1}{\frac{y^2+x^2}{y^2}}\left(\frac{y \cdot 0 - x \cdot 1}{y^2}\right) = \frac{y^2}{x^2+y^2} \cdot \left(\frac{-x}{y^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{-x}{x^2+y^2}$$ **Ans-**

Q.2 If $x^x y^y z^z = c$ then show that $\frac{\partial^2 z}{\partial x \partial y} = (x \log ex)^{-1}$

when $x = y = z$, here z is function of x and y.

Ans- $x^x y^y z^z = c$

Take log both sides

$$\log c = \log x^x + \log y^y + \log z^z$$

$$\log c = x\log x + y\log y + z\log z \quad \text{—(1)}$$

Diff. (1) partically w.r.t. 'x', we get

$$0 = \frac{\partial}{\partial x}\left(x\log x + y\log y + z\log z\right)$$

$$0 = x \cdot \frac{1}{x} + \log x \frac{\partial x}{\partial x} + 0 + z \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x} + \log z \cdot \frac{\partial z}{\partial x}$$

$$1 + \log x + (1+\log z)\frac{\partial z}{\partial x} = 0 \quad \text{—(2)}$$

$$\frac{\partial z}{\partial x}(1+\log z) = -(1+\log x)$$

$$\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{(1+\log z)} \quad -(3)$$

Again diff. (1) partically w.r.to 'y', we have

$$y\cdot\frac{1}{y} + \log y\cdot 1 + z\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial y} + \log z\cdot\frac{\partial z}{\partial y} = 0$$

$$(1+\log y) + (1+\log z)\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{(1+\log z)} \quad -(4)$$

diff.(4) partically w.r.to 'x', we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[\frac{-(1+\log y)}{(1+\log z)}\right]$$

$$\frac{\partial^2 z}{\partial x\partial y} = -\left[\frac{(1+\log z)\frac{\partial}{\partial x}(1+\log y) - (1+\log y)\frac{\partial}{\partial x}(1+\log z)}{(1+\log z)^2}\right]$$

$$\frac{\partial^2 z}{\partial x\partial y} = -\left[\frac{(1+\log z)\cdot 0 - (1+\log y)\left(\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right)}{(1+\log z)^2}\right]$$

$$\frac{\partial^2 z}{\partial x\partial y} = +\frac{(1+\log y)}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}$$

Keep the value of $\partial z/\partial x$ to eq^n -(3) :-

$$\frac{\partial^2 z}{\partial x\partial y} = +\frac{(1+\log y)}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\left(-\frac{(1+\log x)}{(1+\log z)}\right)$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{-1}{z}\,\frac{(1+\log y)(1+\log x)}{(1+\log z)^3}$$

According to Question $x=y=z$

Page No. :
Date :

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{x} \cdot \frac{(1+\log x)^2}{(1+\log x)^3} = \frac{-1}{x(1+\log x)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{x(\log e + \log x)} = \frac{-1}{x(\log ex)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1} \quad \text{(H.P.)}$$

16/8/

Imp
Q.3 If $u = \log(x^3+y^3+z^3-3xyz)$ show that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Ans-(i) $u = \log(x^3+y^3+z^3-3xyz)$ —(1)

Diff. partically w.r.t. x

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \cdot \frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \cdot (3x^2-3yz) \quad \text{—(2)}$$

Diff.(1) partically w.r.t. 'y', we get

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} (3y^2-3xz) \quad \text{—(3)}$$

Diff.(1) partically w.r.t. 'z', we get

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} (3z^2-3xy) \quad \text{—(4)}$$

Addition of eqn 1, 2 & 3

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\cancel{(x^2+y^2+z^2-xy-yz-xz)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-xz)}}$$

$$= \frac{3}{(x+y+z)} \quad \text{(H.P.)} \quad \text{—(5)}$$

POPU

Page No. :
Date :

(ii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) u$

$= \left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\left(\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$

$+ \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$

To eq^n (5)

$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$

$= \frac{-3}{(x+y+z)^2}\cdot(1+0+0) + \frac{(-3)}{(x+y+z)^2}\cdot(0+1+0) + \frac{(-3)}{(x+y+z)^2}(0+0+1)$

$= \frac{-9}{(x+y+z)^2}$ (H.P.)

Q.4 If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$ then prove that

$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$

Solution:- $\because$ u is function of x & y, then

$u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$ —— (1)

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)\right)$ $\left[\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}\right]$

$\frac{\partial u}{\partial y} = \frac{x}{1+\frac{x^2y^2}{1+x^2+y^2}} \cdot x \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{1+x^2+y^2}}\right)$

$$\frac{\partial u}{\partial y} = \frac{x}{\frac{1+x^2+y^2+x^2y^2}{1+x^2+y^2}} \cdot \left[\frac{\sqrt{1+x^2+y^2}\,\frac{\partial y}{\partial y} - y\,\frac{\partial\left(\sqrt{1+x^2+y^2}\right)}{\partial y}}{\left(\sqrt{1+x^2+y^2}\right)^2} \right]$$

$$\frac{\partial u}{\partial y} = \frac{x\cdot\cancel{(1+x^2+y^2)}}{1+x^2+y^2+x^2y^2} \left[\frac{\sqrt{1+x^2+y^2} - y\cdot\frac{1}{\cancel{2}\sqrt{1+x^2+y^2}}\cdot\cancel{2}y}{\cancel{(1+x^2+y^2)}} \right]$$

$$\frac{\partial u}{\partial y} = \frac{x}{1+x^2+y^2+x^2y^2} \left[\frac{1+x^2+\cancel{y^2}-\cancel{y^2}}{\sqrt{1+x^2+y^2}} \right]$$

$$\frac{\partial u}{\partial y} = \frac{x}{(1+x^2)+y^2(1+x^2)} \left(\frac{1+x^2}{\sqrt{1+x^2+y^2}} \right)$$

$$\frac{\partial u}{\partial y} = \frac{x}{(1+y^2)\cancel{(1+x^2)}} \cdot \frac{\cancel{(1+x^2)}}{\sqrt{1+x^2+y^2}} = \frac{1}{1+y^2}\cdot\frac{x}{\sqrt{1+x^2+y^2}} \quad \text{--- (2)}$$

Diff. (2) partically w.r.t. x

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{1+y^2}\,\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{1+x^2+y^2}} \right)$$

$$= \frac{1}{1+y^2} \left(\frac{\sqrt{1+x^2+y^2}\,\frac{\partial x}{\partial x} - x\,\frac{\partial\left(\sqrt{1+x^2+y^2}\right)}{\partial x}}{\left(\sqrt{1+x^2+y^2}\right)^2} \right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{1+y^2} \cdot \left(\frac{\sqrt{1+x^2+y^2} - x \times \frac{1}{\cancel{2}\sqrt{1+x^2+y^2}} \times \cancel{2}x}{(1+x^2+y^2)} \right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{1+y^2} \left(\frac{1+\cancel{x^2}+y^2-\cancel{x^2}}{(1+x^2+y^2)^{3/2}} \right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{\cancel{1+y^2}} \times \frac{\cancel{1+y^2}}{(1+x^2+y^2)^{3/2}} = \frac{1}{(1+x^2+y^2)^{3/2}}$$

Imp

Q.3 (iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-3}{(x+y+z)^2}$ (RTU-2006)

Ans- $\because \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}$ — (2)

$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$ — (3)

$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$ — (4)

we know that

$x^3+y^3+z^3-3xyz = (x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)$

$\{1+\omega+\omega^2=0\}$
$\{\omega^3=1\}$
$\omega = \frac{-1+i\sqrt{3}}{2}$
$\omega^2 = \frac{-1-i\sqrt{3}}{2}$
$\{\omega^4=\omega\}$

$u = \log[(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)]$ — (i)

$\frac{\partial u}{\partial x} = \frac{1}{x+y+z} + \frac{1}{x+y\omega+z\omega^2} + \frac{1}{x+y\omega^2+z\omega}$

$\frac{\partial^2 u}{\partial x^2} = \frac{-1}{(x+y+z)^2} - \frac{1}{(x+y\omega+z\omega^2)^2} - \frac{1}{(x+y\omega^2+z\omega)^2}$ — (ii)

Again from -(i)

$\frac{\partial u}{\partial y} = \frac{1}{x+y+z} + \frac{\omega}{(x+y\omega+z\omega^2)} + \frac{\omega^2}{x+y\omega^2+z\omega}$

$\frac{\partial^2 u}{\partial y^2} = \frac{-1}{(x+y+z)^2} - \frac{\omega^2}{(x+y\omega+z\omega^2)^2} - \frac{\omega^4}{(x+y\omega^2+z\omega)^2}$

$\frac{\partial^2 u}{\partial y^2} = \frac{-1}{(x+y+z)^2} - \frac{\omega^2}{(x+y\omega+z\omega^2)^2} - \frac{\omega}{(x+y\omega^2+z\omega)^2}$ -(iii)

Similar Again from -(i)

$\frac{\partial^2 u}{\partial z^2} = \frac{-1}{(x+y+z)^2} - \frac{\omega}{(x+y\omega+z\omega^2)^2} - \frac{\omega^2}{(x+y\omega^2+z\omega)^2}$ -(iv)

Addition eqn - ii, iii, (iv)

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-3}{(x+y+z)^2} - \left[\frac{1+\omega+\omega^2}{(x+y\omega+z\omega^2)^2}\right]$

$- \left[\frac{1+\omega+\omega^2}{(x+y\omega^2+z\omega)^2}\right]$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-3}{(x+y+z)^2}$$

$(\because 1+\omega+\omega^2=0)$

2018/1/

Q. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, then show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right]$$

Solution:- Here u is function of x, y & z

we ~~have~~ $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$ —— (1)

Diff. (1) partically w.r.t. 'x', we have

$$\frac{1}{a^2+u}(2x) + x^2\left\{\frac{(-1)}{(a^2+u)^2}\right\}\cdot\frac{\partial u}{\partial x} + y^2\left\{\frac{(-1)}{(b^2+u)^2}\cdot\frac{\partial u}{\partial x}\right\}$$

$$+ z^2\left\{\frac{(-1)}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x} = 0$$

$$\frac{2x}{a^2+u} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x}$$

$$\frac{2x}{a^2+u} = \lambda\frac{\partial u}{\partial x} \quad \left\{\because \text{ where } \lambda = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right.$$

$$\frac{\partial u}{\partial x} = \frac{1}{\lambda}\cdot\frac{2x}{a^2+u} \quad \text{—— (2)}$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{1}{\lambda}\frac{2y}{b^2+u} \quad \text{—— (3)}$$

Similarly $\frac{\partial u}{\partial z} = \frac{1}{\lambda}\cdot\frac{2z}{c^2+u}$ —— (4)

Addition of eq^n $(2)^2+(3)^2+(4)^2$ with ~~square of this~~

POPU

Page No.:
Date:

LHS $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2$

$$=\left(\frac{1}{d}\cdot\frac{2x}{a^2+u}\right)^2+\left(\frac{1}{d}\cdot\frac{2y}{b^2+u}\right)^2+\left(\frac{1}{d}\cdot\frac{2z}{c^2+u}\right)^2$$

$$=\frac{4x^2}{d^2(a^2+u)^2}+\frac{4y^2}{d^2(b^2+u)^2}+\frac{4z^2}{d^2(c^2+u)^2} \quad \text{---}$$

RHS $2\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right]$

$$=2\left[x\left(\frac{2x}{d(a^2+u)}\right)+y\left\{\frac{2y}{d(b^2+u)}\right\}+z\left\{\frac{2z}{d(c^2+u)}\right\}\right]$$

in LHS :-

$$\frac{4}{d^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=\frac{4}{\cancel{d^2}}\times\cancel{d}$$

$$=\frac{4}{d} \quad \text{--- (5)}$$

in RHS :-

$$=2\left[\frac{2}{d}\left\{\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{b^2+u}\right\}\right]$$

According to question, ~~wh~~ we have

$$=\frac{4}{d}(1)=\frac{4}{d} \quad \text{--- (6)}$$

$$\boxed{LHS = RHS}$$

Using eqn (5)

$$2\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right]=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2$$

Q. If $\theta = t^n e^{-r^2/4t}$, then what value of n

$$\frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left[r^2 \cdot \frac{\partial \theta}{\partial r} \right) = \frac{\partial \theta}{\partial t}$$

Ans - $\theta = t^n e^{-r^2/4t}$ —— ①

Diff. (1) partial w.r.t. 'r', we have

$$\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t} \cdot \frac{1}{4t} (-2r)$$

$$\frac{\partial \theta}{\partial r} = t^n \left(\frac{-2r}{4t} \right) e^{-r^2/4t} \text{ —— ②}$$

Diff. (1) partial w.r.t. 't', we have

$$\frac{\partial \theta}{\partial t} = n t^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t} \left(\frac{-r^2}{4} \times \frac{-1}{t^2} \right)$$

$$\frac{\partial \theta}{\partial t} = n t^{n-1} e^{-r^2/4t} + \frac{t^n r^2}{4t^2} e^{-r^2/4t}$$

$$\frac{\partial \theta}{\partial t} = \left(n t^{n-1} + \frac{t^n r^2}{4t^2} \right) e^{-r^2/4t} \text{ —— ③}$$

In eqn – ②

$$\frac{\partial \theta}{\partial r} = \frac{-t^{n-1} r}{2} e^{-r^2/4t}$$

$$\Rightarrow r^2 \frac{\partial \theta}{\partial r} = \frac{-r^3 t^{n-1}}{2} e^{-r^2/4t}$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r} \right) = -\frac{t^{n-1}}{2} \frac{\partial}{\partial r}\left[r^3 \cdot e^{-r^2/4t} \right]$$

$$= -\frac{t^{n-1}}{2}\left[r^3 e^{-r^2/4t} \left(\frac{-2r}{4t} \right) + e^{-r^2/4t} \cdot 3r^2 \right]$$

$$= \frac{-t^{n-1}}{2}\left[\left(\frac{-r^4}{2t} + 3r^2 \right) e^{-r^2/4t} \right]$$

$$= \frac{-t^{n-1}}{2} r^2 \left[\left(\frac{-r^2}{2t} + 3 \right) e^{-r^2/4t} \right]$$

$$\Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial \theta}{\partial r} \right) = \frac{1}{\cancel{r^2}} \left(\frac{-t^{n-1}}{2} \right) \cancel{r^2} \left[\left(\frac{-r^2}{2t} + 3 \right) e^{-r^2/4t} \right]$$

Date :

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{t^{n-1}}{2}\left(\frac{-r^2}{2t}+3\right)e^{-r^2/4t} \quad — (4)$$

In eq^n — (3)

$$\frac{\partial\theta}{\partial t} = \left(nt^{n-1} + \frac{t^{n-2}r^2}{4}\right)e^{-r^2/4t} \quad — (5)$$

~~To comparision eq^n (4) & (5)~~

$$\boxed{\cancel{n = -\frac{3}{2}}}$$

According to condition

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$$

$$-\cancel{t^{n-1}}\left(\frac{3}{2}-\frac{r^2}{4t}\right)\cancel{e^{-r^2/4t}} = \cancel{t^{n-1}}\left(n+\frac{r^2}{4t}\right)\cancel{e^{-r^2/4t}}$$

$$\cancel{\frac{r^2}{4t}} - \frac{3}{2} = \cancel{\frac{r^2}{4t}} + n$$

$$\boxed{n = -\frac{3}{2}}$$

21/8/

Homogeneous function :- An expression

$a_0x^ny^0 + a_1x^{n-1}y + a_2x^{n-2}y^2 + a_3x^{n-3}y^3 + \dots\dots a_nx^0y^n$

in which every term is of degree n is called Homogeneous function.

example :- $f(x,y) = \dfrac{x^3+y^3}{x+y}$

$$= \frac{x^3\left(1+\left(\frac{y}{x}\right)^3\right)}{x\left(1+\frac{y}{x}\right)} = x^2\,\phi(y/x)$$

Standard form of Homogeneous function :-

$$f(x,y) = x^n \phi\left(\frac{y}{x}\right)$$

where n is degree of homogeneous function.

Euler Theorem for Homogeneous Function $\Rightarrow$

Statement :- If $f(x,y)$ be the Homogeneous function of degree n, then

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf}$$

Q. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, then show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u(1-4\sin^2 u)$

Ans - we have $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$$\tan u = \frac{x^3+y^3}{x+y} = f \text{ (say)}$$

$$f = \frac{x^3+y^3}{x+y} = \frac{x^3\left(1+\left(\frac{y}{x}\right)^3\right)}{x\left(1+\frac{y}{x}\right)}$$

$$f = x^2\phi(y/x)$$

which is homogeneous function of degree n = 2

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\sin u}{\cos u} \times \cos^2 u$$

$[\because \sin 2x = 2 \sin x \cos x]$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u \quad \text{---} (1)$$

(ii) Diff. (1) partially w.r.t. 'x', we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} + 0 = 2 \cos 2u \cdot \frac{\partial u}{\partial x}$$

multiple to 'x' both side

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \partial y} = 2x \cos 2u \cdot \frac{\partial u}{\partial x} \quad \text{---} (3)$$

Again diff. (1) partially w.r.t. 'y', we obtain

$$x \frac{\partial^2 u}{\partial y \partial x} + 0 + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2 \cos 2u \cdot \frac{\partial u}{\partial y}$$

multiple to 'y' both side

$$xy \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} = 2y \cos 2u \cdot \frac{\partial u}{\partial y} \quad \text{---} (4)$$

Addition eq^n – (3) + (4)

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial y\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$
$$= 2\cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \quad \left(\because \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}\right)$$
from eq^n-(1) $\rightarrow$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} = (2\cos 2u - 1)(\sin 2u)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} = (2(1-2\sin^2 u)-1)(\sin 2u)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)(\sin 2u)$$
(H.P.)

Q. $U = \sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\tan u$

A:- we have $U = \sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$

$$f(x,y) = \sin U = \frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}} = \frac{x^{1/4}(1+(y/x)^{1/4})}{x^{1/5}(1+(y/x)^{1/5})}$$

$$\sin u = x^{1/20}\frac{(1+(y/x)^{1/4})}{(1+(y/x)^{1/5})} = x^{1/20}\phi(y/x)$$

which is homogeneous function of degree $n = 1/20$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = \frac{1}{20}\sin u$$

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)(\cos u) = \frac{1}{20}\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\tan u$$

Q. $U = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{2}\cot$

Ans -

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f(\text{say})$$

$$\cos u = \frac{x(1+y/x)}{x^{1/2}(1+(y/x)^{1/2})} = \frac{x^{1/2}(1+y/x)}{(1+(y/x)^{1/2})}$$

$$\cos u = x^{1/2}\, \phi(y/x)$$

Which is homogeneous function of degree $n = 1/2$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial(\cos u)}{\partial x} + y\frac{\partial(\cos u)}{\partial y} = \frac{1}{2}\cos u$$

$$-x\sin u\frac{\partial u}{\partial x} - y\sin u\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{-1}{2}\frac{\cos u}{\sin u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{2}\cot u$$

23

Q. If $v = x f(y/x) + g(y/x)$ prove that

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 0$$

POPU

Page No.:
Date:

Solution:- We have $u = x f(y/x) + g(y/x)$

$u = v + w$ ---- (1)

where $v = x^1 f(y/x)$ and $w = x^0 g(y/x)$

Clearly v & w are the homogenous function of degree 1 and 0 respectively, then

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1\cdot v \quad \text{——— (2)}$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0\cdot W = 0 \quad \text{——— (3)}$$

additing (2) + (3)

$$x\frac{\partial v}{\partial x} + x\frac{\partial w}{\partial x} + y\frac{\partial v}{\partial y} + y\frac{\partial w}{\partial y} = v + 0$$

$$x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w) = v$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = v \quad \text{——— (4)}$$

Diff. (4) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial v}{\partial x}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = x\frac{\partial v}{\partial x} \quad \text{——— (5)}$$

again Diff. (4) partially w.r.t. 'y', we get

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}$$

$$xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = y\frac{\partial v}{\partial y} \quad \text{——— (6)}$$

additing (5) + (6)

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial y\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

Using ~~from~~ eq^n – ② & ④

$$\frac{x\partial v}{\partial x} + \frac{y\partial v}{\partial x} = v \quad \& \quad \frac{x\partial u}{\partial x} + y\frac{\partial u}{\partial y} = v$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + v = v$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

(H.P.)

Q. Verify euler's Theorem for the function
$f(x,y) = (\sqrt{x}+\sqrt{y})(x^n+y^n)$

Solution:- $f(x,y) = (\sqrt{x}+\sqrt{y})(x^n+y^n)$ —— ①

$$= \sqrt{x}\left(1+(y/x)^{1/2}\right)(x^n)\left(1+(y/x)^n\right)$$

$$= x^{n+1/2}\left(1+(y/x)^{1/2}\right)\left(1+(y/x)^n\right)$$

$$= x^{n+1/2}\,\phi(y/x)$$

which is homogeneous function of degree $n+1/2$, then

$$\frac{x\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \left(n+\frac{1}{2}\right)f \quad \text{—— ②}$$

Now from eq^n –①

$$\frac{\partial f}{\partial x} = (\sqrt{x}+\sqrt{y})\frac{\partial(x^n+y^n)}{\partial x} + (x^n+y^n)\frac{\partial}{\partial x}(\sqrt{x}+\sqrt{y})$$

$$= (\sqrt{x}+\sqrt{y})(nx^{n-1}) + (x^n+y^n)\left(\frac{1}{2}x^{-1/2}\right)$$

$$x\frac{\partial f}{\partial x} = x\cdot(nx^{n-1})(\sqrt{x}+\sqrt{y}) + \frac{x\cdot}{2x^{1/2}}(x^n+y^n)$$

POPU

Page No.:
Date: / /

$$= nx^n(\sqrt{x}+\sqrt{y}) + \frac{\sqrt{x}}{2}(x^n+y^n) = x\frac{\partial f}{\partial x} \quad \text{--- (3)}$$

then again from eq^n - (1)

$$\frac{\partial f}{\partial y} = (\sqrt{x}+\sqrt{y})\frac{\partial(x^n+y^n)}{\partial y} + (x^n+y^n)\frac{\partial(\sqrt{x}+\sqrt{y})}{\partial y}$$

$$= (\sqrt{x}+\sqrt{y})(ny^{n-1}) + (x^n+y^n)\left(\frac{1}{2\sqrt{y}}\right)$$

$$y\frac{\partial f}{\partial y} = y \cdot n(y^{n-1})(\sqrt{x}+\sqrt{y}) + y \cdot \frac{1}{2\sqrt{y}}(x^n+y^n)$$

$$y\frac{\partial f}{\partial y} = ny^n(\sqrt{x}+\sqrt{y}) + \frac{\sqrt{y}}{2}(x^n+y^n) \quad \text{--- (4)}$$

addition (3) + (4)

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = (\sqrt{x}+\sqrt{y})(n)(x^n+y^n) + \frac{(\sqrt{x}+\sqrt{y})(x^n+y^n)}{2}$$

$$= (\sqrt{x}+\sqrt{y})(x^n+y^n)\left(n+\frac{1}{2}\right)$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \left(n+\frac{1}{2}\right)f \quad \text{--- (5)}$$

from eq^n - (2) and (5), Euler's ~~verify~~ theorem is verify.

Total derivative ⇒ (1) if $u = f(x,y)$ and $x = \phi(t)$, $y = \phi(t)$, then total derivative of given function is

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

or

$$\boxed{du = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy}$$

POPU
Page No. :
Date :

(2) If $u = f(x,y)$ and $x = \phi(y)$

$$\frac{du}{dy} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dy} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dy}$$

$$\boxed{\frac{du}{dy} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dy} + \frac{\partial u}{\partial y}}$$

(3) If $u = f(x,y)$ and $y = \phi(x)$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$\boxed{\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}}$$

(4) Special Case :-

If $u = \text{Constant}$

$$\frac{du}{dx} = 0$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$-\partial u/dx = +\partial u/\partial y \cdot \frac{dy}{\partial x}$$

$$\boxed{\frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = \frac{-p}{q}}$$

(5) Chain Rule (Change of variable)

If $u = f(x,y)$ and $x = \phi(t_1, t_2)$, $y = \phi(t_1, t_2)$

$$\boxed{\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1}}$$ (t_2 is keeping as constant)

$$\boxed{\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}}$$ ($\because t_1$ is keeping as Constant)

(6) $$\frac{d^2y}{dx^2} = \frac{-\left[\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\cdot\left(\frac{\partial f}{\partial x}\right)^2\right]}{\left(\frac{\partial f}{\partial y}\right)^3}$$

26/8/

Q. If $u = f(y-z, z-x, x-y)$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Solve:- let $t_1 = y-z$ | $t_2 = z-x$ | $t_3 = x-y$

$\frac{\partial t_1}{\partial x} = 0$ | $\frac{\partial t_2}{\partial x} = -1$ | $\frac{\partial t_3}{\partial x} = 1$

$\frac{\partial t_1}{\partial y} = 1$ | $\frac{\partial t_2}{\partial y} = 0$ | $\frac{\partial t_3}{\partial y} = -1$

$\frac{\partial t_1}{\partial z} = -1$ | $\frac{\partial t_2}{\partial z} = 1$ | $\frac{\partial t_3}{\partial z} = 0$

$\cancel{\frac{\partial y}{\partial x}}$ $u = f(y-z, z-x, x-y)$

$u = f(t_1, t_2, t_3)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial x}$$

$$\cancel{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial y}}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}(0) + \frac{\partial u}{\partial t_2}(-1) + \frac{\partial u}{\partial t_3}(1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_3} - \frac{\partial u}{\partial t_2} \quad \text{——— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}(1) + \frac{\partial u}{\partial t_2}(0) + \frac{\partial u}{\partial t_3}(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1} - \frac{\partial u}{\partial t_3} \quad \text{——} \quad (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial z} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial z} + \frac{\partial u}{\partial t_3} \cdot \frac{\partial t_3}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t_1}(-1) + \frac{\partial u}{\partial t_2}(1) + \frac{\partial u}{\partial t_3}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t_2} - \frac{\partial u}{\partial t_1} \quad \text{——} \quad (3)$$

Addition eq^n — 1, 2, 3

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$$

Q. $x^y + y^x = c$ then find dy/dx

Ans- $\because \quad \dfrac{dy}{dx} = \dfrac{-\partial u/\partial x}{\partial u/\partial y}$

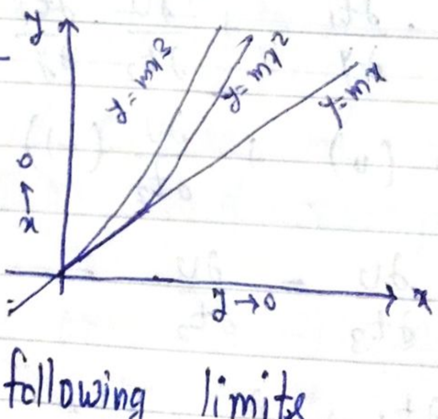
we have $x^y + y^x = c$

~~$y \log x + x \log y = \log c$~~

~~partial~~

$$\frac{dy}{dx} = -\frac{(y x^{y-1} + y^x \log_e y)}{(x^y \log_e x + x y^{x-1})}$$

Continuity:-



Q.1 Evaluate the following limits

(i) $\lim\limits_{\substack{x \to 2 \\ y \to 3}} \dfrac{x^2+y^3}{2x^2y}$

(ii) $\lim\limits_{\substack{x \to 1 \\ y \to 2}} \dfrac{x^2+y}{3x+y^2}$

Ans-(i) $\lim\limits_{\substack{x\to 2\\ y\to 3}} \dfrac{x^2+y^3}{2x^2y} = \lim\limits_{x\to 2}\left\{\lim\limits_{y\to 3} \dfrac{x^2+(3)^3}{2x^2(3)}\right\} = \lim\limits_{x\to 2} \dfrac{x^2+27}{6x^2}$

$= \dfrac{(2)^2+27}{6\times(2)^2} = \dfrac{31}{24}$

(ii) $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{x^2+y}{3x+y^2} = \lim\limits_{x\to 1}\left\{\lim\limits_{y\to 2} \dfrac{x^2+y}{3x+y^2}\right\} = \lim\limits_{x\to 1}\left(\dfrac{x^2+2}{3x+4}\right)$

$= \dfrac{(1)^2+2}{3(1)+4} = \dfrac{2+1}{7} = \dfrac{3}{7}$

Q. Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

Ans- $\lim\limits_{x\to 0}\left\{\lim\limits_{y\to 0} \dfrac{x^2y}{x^4+y^2}\right\} = \lim\limits_{x\to 0}\left(\dfrac{x^2(0)}{x^4}\right) = \lim\limits_{x\to 0}\left(\dfrac{0}{x^4}\right)$

$\lim\limits_{x\to 0} 0 = 0$

Again $\lim\limits_{y\to 0}\left\{\lim\limits_{x\to 0} \dfrac{x^2y}{x^4+y^2}\right\} = \lim\limits_{y\to 0}\left(\dfrac{0(y)}{0+y^2}\right) = \lim\limits_{y\to 0}\left(\dfrac{0}{y^2}\right)$

$\lim\limits_{y\to 0}\{0\} = 0$

Now along path $y = mx$

$\lim\limits_{\substack{y\to mx\\ x\to 0}} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{x\to 0}\left\{\lim\limits_{y\to mx} \dfrac{x^2y}{x^4+y^2}\right\}$

$= \lim\limits_{x\to 0} \dfrac{mx^3}{x^4+m^2x^2} = \lim\limits_{x\to 0} \dfrac{mx^3}{x^2(x^2+m^2)} = \dfrac{0}{m^2}$

$= 0$

Along path $y = mx^2$

$\lim\limits_{\substack{y\to mx^2\\ x\to 0}} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{x\to 0}\left\{\lim\limits_{y\to mx^2} \dfrac{x^2y}{x^4+y^2}\right\} = \lim\limits_{x\to 0} \dfrac{m\cancel{x^4}}{\cancel{x^4}(1+m^2)}$

$= \dfrac{m}{1+m^2}$

Since the limit along ~~Now in~~ $y = mx^2$ is depend on 'm', where m is a number

$\therefore$ limit does not exist

Q. Discuss the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}}, & x,y \neq 0 \\ 2 & x,y = 0 \end{cases}$$ at the origin.

Ans- We have $f(x,y) = \dfrac{x^2}{\sqrt{x^2+y^2}}$

$$LHL = \lim_{x\to 0}\left\{\lim_{y\to 0}\left(\frac{x^2}{\sqrt{x^2+y^2}}\right)\right\} = \lim_{x\to 0}\left(\frac{x^2}{x}\right) = \lim_{x\to 0} x$$
$$= 0$$

$$RHL = \lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+y^2}}\right\} = \lim_{y\to 0}\frac{0}{y^2} = \lim_{y\to 0} 0$$
$$= 0$$

Along $y = mx$

$$\lim_{\substack{y\to mx\\ x\to 0}}\left\{\frac{x^2}{\sqrt{x^2+y^2}}\right\} = \lim_{x\to 0}\left\{\frac{x^2}{\sqrt{x^2+m^2x^2}}\right\}$$
$$= \lim_{x\to 0}\left\{\frac{x}{\sqrt{1+m^2}}\right\} = \frac{0}{\sqrt{1+m^2}} = 0$$

Along path $y = mx^2$

$$\lim_{\substack{y\to mx^2\\ x\to 0}}\left(\frac{x^2}{\sqrt{x^2+y^2}}\right) = \lim_{x\to 0}\left(\frac{x^2}{\sqrt{x^2+m^2x^4}}\right) = \lim_{x\to 0}\left(\frac{x^2}{x\sqrt{1+m^2x^2}}\right)$$
$$= \frac{0}{\sqrt{1+0}} = 0$$

limit along any path is ~~since~~ same there for $(\because)$ limit exist value of limit is zero at origin

function $\lim_{\substack{x\to 0\\ y\to 0}}\left\{\dfrac{x^2}{\sqrt{x^2+y^2}}\right\} = 0 \neq 2 = f(0,0)$

function is not continuous at $x=0$.

Directional Derivative ⇒ * Let $f(x,y)$ be the function of two variable, then its partial derivative at the point (x_0,y_0) is defined as

Rate of change

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

and

$$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$ where

f_x and f_y represent rates of change in f in x and y - directions respectively.

* Let $f(x,y)$ be the function of two variables, then its directional derivative at (x_0,y_0) is denoted by $D_u(x_0,y_0)$, in the direction of Unit vector $\bar{u} = \langle a,b \rangle$ is defined as.

$$D_u f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h}$$

* Theorem (Statement) :- If f is the function of two variables, then direction derivatives of in the direction $\bar{U} = \langle a,b \rangle$ is given by

Imp!

$$D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

Note:- If $\bar{u}$ makes an angle θ in positive direction, then directional derivative

$$D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

29/8/

Q.1 Find the directional derivative of the function at the given point P in the direction of the vector $\vec{u}$

$$f(x,y) = 1 + 2x\sqrt{y}, \quad P = (3,4), \quad \vec{u} = \langle 4, -3 \rangle$$

Ans- Here $x_0 = 3, \quad y_0 = 4$

Unit vectors in direction of $\bar{u} = \left\langle \frac{4}{\sqrt{4^2+(-3)^2}}, \frac{-3}{\sqrt{4^2+(-3)^2}} \right\rangle$

$$\left\langle \frac{4}{5}, \frac{-3}{5} \right\rangle = \langle a, b \rangle$$

Directional derivative:-

$$D_u f(x_0 y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

$$D_u f(3,4) = \lim_{h \to 0} \frac{f(3 + \frac{4}{5}h, 4 - \frac{3}{5}h) - f(3,4)}{h}$$

$$D_u f(3,4) = \lim_{h \to 0} \left[\frac{\left(1 + 2(3 + \frac{4}{5}h)\sqrt{4 - \frac{3}{5}h}\right) - \left(1 + 2 \times 3\sqrt{4}\right)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\cancel{1} + (6 + \frac{8}{5}h)(\sqrt{4 - \frac{3}{5}h}) - \cancel{1} - 12}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{(6 + \frac{8}{5}h)(\sqrt{4 - \frac{3}{5}h}) - 12}{h} \right] \quad \left(\frac{0}{0} \text{ form}\right)$$

By L'Hospital form:-

$$\lim_{h \to 0} \frac{\frac{d}{dh}\left((6 + \frac{8}{5}h)(\sqrt{4 - \frac{3}{5}h}) - 12 \right)}{\frac{dh}{dh}}$$

$$\lim_{h \to 0} \left(6 + \frac{8}{5}h\right) \times \frac{1}{2\sqrt{4 - \frac{3}{5}h}} \left(\frac{-3}{5}\right) + \sqrt{4 - \frac{3}{5}h} \left(\frac{8}{5}\right)$$

$$\lim_{h\to 0} \frac{8}{5}\sqrt{4-\frac{3}{5}h} - \frac{3}{5}\times\frac{1}{2}\left(6+\frac{8}{5}h\right)\times\frac{1}{\sqrt{4-\frac{3}{5}h}}$$

$$\lim_{h\to 0} \frac{\frac{8}{5}\left(4-\frac{3}{5}h\right) - \frac{3}{10}\left(6+8/5h\right)}{\sqrt{\left(4-3/5h\right)}} = \frac{\frac{32}{5}-\frac{18}{10}}{\sqrt{4}} = \frac{\frac{64-18}{10}}{\sqrt{4}}$$

$$23\,\frac{\cancel{46}}{10}\times\frac{1}{\cancel{2}}$$

$$= \frac{23}{10}$$

Q.2 Find the directional derivative of $f(x,y)$ at the given pt P in the direction indicated by the angle θ

$f(x,y) = x^2y^3 - y^4$, $P = (2,1)$, $\theta = \frac{\pi}{4}$

Solution Given :- $f(x,y) = x^2y^3 - y^4$

$P = (2,1) = (x_0, y_0)$, $\theta = \pi/4$

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

$$f_x(2,1) = \lim_{h\to 0} \frac{f(2+h, 1) - f(2,1)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2 - \cancel{1} - (2)^2 + \cancel{1}}{h} =$$

$$= \lim_{h\to 0} \frac{(4+h)\cancel{h}}{\cancel{h}} = 4$$

$$f_y(2,1) = \lim_{h\to 0} \frac{f(2,1+h) - f(2,1)}{h}$$

$$= \lim_{h\to 0} \frac{(2)^2(1+h)^3 - (1+h)^4 - 4 + 1}{h}$$

$$= \lim_{h\to 0} \frac{4(1+h)^3 - (1+h)^4 - 3}{h} = \cancel{(1+h)^3[4-1-h]-3}$$

By D'L Hospital form :-

$$= \lim_{h\to 0} \frac{12(1+h)^2 - 4(1+h)^3}{1} = 12 - 4 = 8$$

$$\left\{ \begin{array}{l} \vec{\nabla} f \to \text{Gradient} \\ \vec{\nabla} \cdot \vec{f} \to \text{divergence} \\ \vec{\nabla} \times \vec{f} \to \text{Curl} \end{array} \right\}$$

$$D_u f(x_0, y_0) = 4 \cos\frac{\pi}{4} + 8 \sin\frac{\pi}{4}$$

$$D_u f(x_0, y_0) = (4+8) \times \frac{1}{\sqrt{2}} = 12 \times \frac{1}{\sqrt{2}} = 6\sqrt{2}$$

Gradient ⇒

let $f(x,y)$ be a scalar function, then Gradient of f is given by

$$\vec{\nabla} f(x,y) = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}$$

$$= \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

If $f(x,y,z)$ be a scalar function of x,y,z then

$$\vec{\nabla} f(x,y,z) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) f$$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Note ⇒ $\boxed{D_u f(x,y) = [\nabla f(x,y)] \cdot \vec{u}}$ → unit vector $\left(\frac{\vec{v}}{|\vec{v}|}\right)$

Q. find gradient of $f(x,y) = 5xy^2 - 4x^3y$ at the point $P(1,2)$

Ans – $\cancel{f(x_0,y_0) = 5x_0 y_0^2 - 4x_0^3 y_0}$

$\cancel{\vec{\nabla} f(x,y) = \hat{i}}$

$$\vec{\nabla} f(x,y) = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}$$

$$\vec{\nabla} f(x,y) = \hat{i}\frac{\partial(5xy^2 - 4x^3y)}{\partial x} + \hat{j}\frac{\partial(5xy^2 - 4x^3y)}{\partial y}$$

$$\vec{\nabla} f(x,y) = \hat{i}(5y^2 - 12x^2y) + \hat{j}(10xy - 4x^3)$$

$$\vec{\nabla} f(1,2) = \hat{i}(5(2)^2 - 12(1)^2(2)) + \hat{j}(10(1)(2) - 4(1)^3)$$

$$= \hat{i}(20 - 24) + \hat{j}(20 - 4)$$

$$\vec{\nabla} f(1,2) = -4\hat{i} + 16\hat{j} = \langle -4, 16 \rangle$$

30/8/

Q. Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $P(2,-1)$ in the direction of $\vec{v} = 2\hat{i} + 5\hat{j}$

Ans-

$$\vec{\nabla} f(x,y) = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}$$

$$= \hat{i}\frac{\partial(x^2y^3 - 4y)}{\partial x} + \hat{j}\frac{\partial(x^2y^3 - 4y)}{\partial y}$$

$$= \hat{i}(2xy^3) + \hat{j}(3y^2x^2 - 4)$$

$$\vec{\nabla} f(2,-1) = \hat{i}(2\cdot 2(-1)^3) + \hat{j}(3(-1)^2(2)^2 - 4)$$

$$= \hat{i}(-4) + \hat{j}(12 - 4)$$

$$-4\hat{i} + 8\hat{j}$$

$\left(\because \text{Unit vector} = \frac{\vec{v}}{|\vec{v}|}\right)$

$$D_u f(x,y) = (\vec{\nabla} f(2,-1)) \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$= (-4\hat{i} + 8\hat{j})(2\hat{i} + 5\hat{j}) / \sqrt{29}$$

$$= \frac{-8 + 40}{\sqrt{29}} = \frac{32}{\sqrt{29}} \quad \text{Ans}$$

Divergence ⇒ let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector point function, then divergence of $\vec{F}$ is defined as

$$\vec{\nabla}\cdot\vec{F} = \text{Div}\,\vec{F}$$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial x} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

Solenoidal:- Vector pt function $\vec{F}$ is said be solenoidal

$$\boxed{\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

$$\boxed{\vec{\nabla}\cdot\vec{F} = 0}$$

Curl ⇒

let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector pt function then, curl of given function is

$$\vec{\nabla}\times\vec{F} = \text{Curl}\,\vec{F}$$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Irrotational:- A vector pt function $\vec{F}$ is said to be irrotational then

$$\boxed{\vec{\nabla}\times\vec{F} = 0}$$

Q. If $\phi = x^3 + y^3 + z^3 - 3xyz$ find (i) $\nabla\phi$ (ii) div$\vec{\nabla}\phi$ at the point $(1,2,3)$.

Sol- We have $\phi = x^3 + y^3 + z^3 - 3xyz$

(i) $$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial(x^3+y^3+z^3-3xyz)}{\partial x} + \hat{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + \hat{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$= \hat{i}(3x^2-3yz) + \hat{j}(3y^2-3xz) + \hat{k}(3z^2-3xy)$

$\vec{\nabla}\phi(1,2,3) = \cancel{\hat{i}(3-3(2)(3)) + \hat{j}(3-3) + \hat{k}(3-3)} \cancel{= 0.\hat{i}+0.\hat{j}+0\hat{k}}$

$= \hat{i}(3-3(2)(3)) + \hat{j}(3\cdot2^2-3\cdot1\cdot3) + \hat{k}(3\cdot3^2-3\cdot1\cdot2)$

$= \hat{i}(3-18) + \hat{j}(12-9) + \hat{k}(27-6)$

$\vec{\nabla}\phi = -15\hat{i} + 3\hat{j} + 21\hat{k}$

(ii) $\text{div}\vec{\nabla}\phi = \vec{\nabla}\cdot(\vec{\nabla}\phi)$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}(3x^2-3yz) + \hat{j}(3y^2-3xz) + \hat{k}(3z^2-3xy)\right)$

$= 6x + 6y + 6z$

$(\text{div}\vec{\nabla}\phi)_{(1,2,3)} = 6(1) + 6(2) + 6(3)$

$= 6 + 12 + 18 = 36$

Q. Find the curl of the vector

$\vec{F} = (z^2+2x+3y)\hat{i} + (3x+2y+z)\hat{j} + (y+2zx)\hat{k}$

Ans- Curl $\vec{F} = \vec{\nabla}\times\vec{F}$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left[(z^2+2x+3y)\hat{i} + (3x+2y+z)\hat{j} + (y+2zx)\hat{k}\right]$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2+2x+3y & 3x+2y+z & y+2zx \end{vmatrix}$$

$= \hat{i}\left(\frac{\partial}{\partial y}(y+2zx) - \frac{\partial}{\partial z}(3x+2y+z)\right) - \hat{j}\left(\frac{\partial}{\partial x}(y+2zx) - \frac{\partial}{\partial z}(z^2+2x+3y)\right)$

$+ \hat{k}\left[\frac{\partial}{\partial x}(3x+2y+z) - \frac{\partial}{\partial y}(z^2+2x+3y)\right]$

$= \hat{i}(1-1) - \hat{j}(2z - 2z) + \hat{k}(3-3)$

$= 0\cdot\hat{i} + 0\cdot\hat{j} + 0\cdot\hat{k}$

Q. A fluid motion is given by $\vec{q} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ Is this motion is irrotational? If so, find the velocity potential (Scalar potential)

Ans -

$$\vec{q} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$$

$$\text{Curl } \vec{q} = \vec{\nabla} \times \vec{q}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y+z) & (z+x) & (x+y) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right] - \hat{j}\left(\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial z}(y+z)\right)$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$$

$$= 0\cdot\hat{i} + 0\cdot\hat{j} + 0\cdot\hat{k}$$

Given vector $\vec{q}$ is irrotational

let ϕ be the velocity potential

$$\vec{q} = \vec{\nabla}\phi$$

$$(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing the coefficient of $\hat{i}, \hat{j}$ & $\hat{k}$, we have

$$\frac{\partial \phi}{\partial x} = y+z \quad , \quad \frac{\partial \phi}{\partial y} = z+x, \quad \frac{\partial \phi}{\partial z} = x+y$$

To total derivative

$$d\phi = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy + \frac{\partial \phi}{\partial z} \cdot dz$$

$$d\phi = (y+z)\,dx + (z+x)\,dy + (x+y)\,dz$$

$$d\phi = y\,dx + z\,dx + z\,dy + x\,dy + x\,dz + y\,dz$$

$$\int d\phi = \int (x\,dy + y\,dx) + (x\,dz + z\,dx) + (y\,dz + z\,dy)$$

$$\int d\phi = \int d(xy) + \int d(xz) + \int d(yz)$$

$$\phi = xy + xz + yz + c$$

Q. Find the values of a, b, c such that

$$\vec{A} = (x+2y+az)\,\hat{i} + (bx-3y-z)\,\hat{j} + (4x+cy+2z)\,\hat{k}$$

is irrotational vector field. Also find its scalar potential.

Ans - $\text{Curl } \vec{A} = \vec{\nabla} \times \vec{A} = 0$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$$

$$\hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0\cdot\hat{i} + 0\cdot\hat{j} + 0\cdot\hat{k}$$

$a = 4$, $b = 2$, $c = -1$

Given vector $\vec{A}$ is irrotational

$$\vec{A} = \vec{\nabla}\phi \qquad \text{(let } \phi \text{ be velocity potential)}$$

$$(x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

$$= \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$x+2y+4z = \frac{\partial \phi}{\partial x}, \quad 2x-3y-z = \frac{\partial \phi}{\partial y}, \quad 4x-y+2z = \frac{\partial \phi}{\partial z}$$

$$d\phi = \frac{\partial \phi}{\partial x}\cdot dx + \frac{\partial \phi}{\partial y}\cdot dy + \frac{\partial \phi}{\partial z}\cdot dz$$

$$d\phi = (x+2y+4z)\,dx + (2x-3y-z)\,dy + (4x-y+2z)\,dz$$

$$d\phi = x\,dx + 2y\,dx + 4z\,dx + 2x\,dy - 3y\,dy - z\,dy + 4x\,dz - y\,dz + 2z\,dz$$

$$\int d\phi = \int x\,dx + 2z\,dz - 3y\,dy + 2(x\,dy + y\,dx) - (y\,dz + z\,dy) + 4(x\,dz + z\,dx)$$

$$\phi = \frac{x^2}{2} + z^2 - \frac{3y^2}{2} + 2xy - yz + 4xz + k$$

1/9/

EQUATION OF TANGENT PLANE and NORMAL LINE ⟹

The tangent plane at the $p(x_0, y_0, z_0)$ on the surface $f(x,y,z) = c$, where f differentiable, is given by

$$\left(\frac{\partial f}{\partial x}\right)_p (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_p (y-y_0) + \left(\frac{\partial f}{\partial z}\right)_p (z-z_0) = 0$$

eqⁿ of Normal

$$x = x_0 + \left(\frac{\partial f}{\partial x}\right)_p t, \quad y = y_0 + \left(\frac{\partial f}{\partial y}\right)_p t, \quad z = z_0 + \left(\frac{\partial f}{\partial z}\right)_p t$$

Page No.
Date :

Q.1 Find the tangent plane and normal line to the surface. $f(x,y,z) = x^2+y^2+z-9=0$ at pt $P(1,2,4)$

Ay- $\left(\frac{\partial f}{\partial x}\right)_p = 2x = 2(1) = 2$, $\left(\frac{\partial f}{\partial y}\right)_p = 2y = 2\times2 = 4$

$\left(\frac{\partial f}{\partial z}\right)_p = 1$

tangent plane :-

$$\left(\frac{\partial f}{\partial x}\right)_p (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_p (y-y_0) + \left(\frac{\partial f}{\partial z}\right)_p (z-z_0) = 0$$

$$(2)(x-1) + (4)(y-2) + (1)(z-4) = 0$$

$$2x-2+4y-8+z-4 = 0$$

$2x+4y+z = 14$ — , which is required plane

eqn of normal

$x = x_0 + \left(\frac{\partial f}{\partial x}\right)_p t$, $y = 2+(4)t$, $z = 4+t$

$x = 1+2t$ $\quad y = 2+4t$

$\frac{x-1}{2} = t$, $\frac{y-2}{4} = t$, $z-4 = t$

$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1}$ which is required eqn of normal.

H.W.

Q.2 Find the tangent plane and normal line to surface $2x^2+y^2+2z=3$ at point $P=(2,1,-3)$

Q.3 Find tangent plane & normal line to surface $f: x^2+2y^2+3z^2=12$ at pt $P(1,2,-1)$

Ay-2 $f = 2x^2+y^2+2z-3$

$\left(\frac{\partial f}{\partial x}\right)_p = 4x = 8$, $\left(\frac{\partial f}{\partial y}\right)_p = 2y = 2$, $\left(\frac{\partial f}{\partial z}\right)_p = 2$

tangent plane

$$\frac{\partial f}{\partial x}(x-2) + \frac{\partial f}{\partial y}(y-1) + \frac{\partial f}{\partial z}(z+3) = 0$$

$8(x-2) + 2(y-1) + 2(z+3) = 0$

$8x + 2y + 2z = 12 \Rightarrow 4x + y + z = 6$, which is rq. plane

eqn of ~~pl~~ NORmal

$x = 2 + 8t$, $y = 1 + 2t$, $z = -3 + 2t$

$\frac{x-2}{8} = t$, $\frac{y-1}{2} = t$, $\frac{z+3}{2} = t$

$\frac{x-2}{8} = \frac{y-1}{2} = \frac{z+3}{2}$ which is required eqn of normal

Qy-3 $f = x^2 + 2y^2 + 3z^2 - 12 = 0$, $P(1, 2, -1)$

$\left(\frac{\partial f}{\partial x}\right)_p = 2x = 2$, $\left(\frac{\partial f}{\partial y}\right)_p = 4y = 8$

$\left(\frac{\partial f}{\partial z}\right)_p = 6z = -6$

tangent plane:-

$\left(\frac{\partial f}{\partial x}\right)_p (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_p (y-y_0) + \left(\frac{\partial f}{\partial z}\right)_p (z-z_0) = 0$

$(2)(x-1) + (8)(y-2) + (-6)(z+1) = 0$

$2x - 2 + 8y - 16 - 6z - 6 = 0$

$2x + 8y - 6z = 24$

$x + 4y - 3z = 12$ which is required eqn of plane.

Eqn of NoRmal:-

$x = x_0 + \left(\frac{\partial f}{\partial x}\right)_p t$, $y = y_0 + \left(\frac{\partial f}{\partial y}\right)_p t$, $z = z_0 + \left(\frac{\partial f}{\partial z}\right)_p t$

POPU

Page No. :
Date : / /

$x = 1 + 2t$, $y = 2 + 8t$, $z = -1 + -6t$

$\frac{x-1}{2} = t$, $\frac{y-2}{8} = t$, $\frac{z+1}{-6} = t$

$\frac{x-1}{2} = \frac{y-2}{8} = \frac{z+1}{-6}$, which is required eqⁿ of normal.

☆ Maxima and Minima ⇒

Let $f(x,y) = c$ be the function of two variables, then we have.

$$\frac{\partial f}{\partial x} = p \ , \ \frac{\partial f}{\partial y} = q \ , \ \frac{\partial^2 f}{\partial x^2} = r \ , \ \frac{\partial^2 f}{\partial x \partial y} = s \ , \ \frac{\partial^2 f}{\partial y^2} = t$$

To determine minima & maxima of given function, we use following steps.

(i) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

(ii) $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, then we get stationary points. For example $(x = x_0, y = y_0)$

(iii) If $rt - s^2 > 0$, $r > 0$, then f will be minimum

(iv) If $rt - s^2 > 0$, $r < 0$, then f will be maximum.

(v) If $rt - s^2 < 0$, then function has saddle pt.

example:- ① Discuss the maximum and minimum of $x^2 + y^2 + 6x + 12$

Ans- $f = x^2 + y^2 + 6x + 12$ — ①

$\frac{\partial f}{\partial x} = 2x + 6$ — ② , $\frac{\partial f}{\partial y} = 2y$ — ③

For maxima & minima $\frac{\partial f}{\partial x} = 0 \Rightarrow 2x + 6 = 0$ $x = -3$

$\frac{\partial f}{\partial y} = 2y \Rightarrow 0$ $y = 0$

Stationary point $(-3, 0)$

At $(-3,0)$ $r = \frac{\partial^2 f}{\partial x^2} = 2$, $s = \frac{\partial^2 f}{\partial x \partial y} = 0$, $t = \frac{\partial^2 f}{\partial y^2} = 2$

$rt - s^2$

$2 \times 2 - 0 = 4 > 0$ and $r > 0 \Rightarrow 2 > 0$

then f ~~is will be~~ minima $rt - s^2 > 0, r > 0$

Min. Value $f(-3,0) = (-3)^2 + (0)^2 + 6(-3) + 12$

$= 9 - 18 + 12 = 3$

Q.2 Show that the min. value of $f(x,y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ is $3a^2$.

Ans-

Given $f(x,y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ — (1)

$\frac{\partial f}{\partial x} = y + a^3\left(\frac{-1}{x^2}\right) = y - \frac{a^3}{x^2}$ — (2)

$\frac{\partial f}{\partial y} = x + a^3\left(\frac{-1}{y^2}\right) = x - \frac{a^3}{y^2}$ — (3)

for minima & maxima

$\frac{\partial f}{\partial x} = 0 \Rightarrow y - \frac{a^3}{x^2} = 0 \Rightarrow y = \frac{a^3}{x^2} \Rightarrow x^2 y = a^3$ — (4)

$\frac{\partial f}{\partial x} = 0 \Rightarrow x - \frac{a^3}{y^2} = 0 \Rightarrow x = \frac{a^3}{y^2}$ — (5)

$xy^2 = a^3 = x^2 y$ (from (4),(5))

$xy^2 - x^2 y = 0$

$xy(y - x) = 0$

$x = 0, y = 0$

$y = x$

take y value in (4)

$x = \frac{a^3}{x^2} \Rightarrow x^3 = a^3$

$x = a = y$

Statinory point (a, a)

$\frac{\partial^2 f}{\partial x^2} = +\frac{2a^3}{x^3}$, $\frac{\partial^2 f}{\partial x \partial y} = 1$, $\frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$

At (a,a)

$r = 2$, $s = 1$, $t = 2$

$rt - s^2 > 0$, $r > 0$

$4 - 1 > 0$, then f is minima

$3 > 0$ at (a,a)

min. value

$f(a,a) = xy + a^3\left(\frac{1}{a}+\frac{1}{a}\right)$

$= a \times a + \frac{a^3}{a}(2) \Rightarrow a^2 + 2a^2$

$= 3a^2$

(H.P.)

Q.3 Find the maximum and minimum values of $f(x,y) = \sin x \sin y \sin(x+y)$

Ans - Given $f(x,y) = \sin x \sin y \sin(x+y)$ —— ①

$\frac{\partial f}{\partial x} = \sin y[\sin(x+y)\cos x + \sin x \cos(x+y)]$

$\frac{\partial f}{\partial x} = \sin y[\sin(x+x+y)] \Rightarrow \sin y \cdot \sin(2x+y)$ —— ②

$\frac{\partial f}{\partial y} = \sin x[\cos y \sin(x+y) + \cos(x+y)\sin y]$

$\frac{\partial f}{\partial y} = \sin x[\sin(x+y+y)] = \sin x \cdot \sin(x+2y)$ —— ③

For maxima & minima

$\frac{\partial f}{\partial x} = \sin y \sin(2x+y) = 0 \Rightarrow$ $y = 0$, $2x+y = 0$ ✓

× $y = \pi$, $2x+y = \pi$ ✓

× $y = 2\pi$, $2x+y = 2\pi$ ✓

Page No. :
Date :

$\frac{\partial f}{\partial y} = \sin x \cdot \sin(x+2y) = 0$

✓ $x=0, \quad x+2y=0$ ✓

× $x=\pi, \quad x+2y=\pi$ ✓

× $x=2\pi, \quad x+2y=2\pi$ ✓

$x=0, y=0 \quad , \quad 2x+y=0, \quad x+2y=0$

$\underbrace{2x+y=0, \quad x+2y=0}_{x=0,\, y=0}$

$\underbrace{x+2y=\pi, \quad 2x+y=\pi}_{x=\frac{\pi}{3},\, y=\frac{\pi}{3}}$ $\qquad$ $\underbrace{x+2y=2\pi, \quad 2x+y=2\pi}_{x=\frac{2\pi}{3},\, y=\frac{2\pi}{3}}$

Statinary point $(0,0), \left(\frac{\pi}{3}, \frac{\pi}{3}\right) \& \left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$

$\frac{\partial^2 f}{\partial x^2} = 2\sin y \cos(2x+y)$, $\frac{\partial^2 f}{\partial x \partial y} = \sin x \cos(x+2y) + \cos x \sin(x+2y)$
$= \sin(2x+2y)$

$\frac{\partial^2 f}{\partial y^2} = 2\sin x \cos(x+2y)$

At $(0,0)$

$r=0, \quad s=0, \quad t=0$

At $(\pi/3, \pi/3)$

$r = 2\sin\frac{\pi}{3}\cos\pi = 2 \times \frac{\sqrt{3}}{2} \times -1 = -\sqrt{3}$

$s = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

$t = 2\sin\frac{\pi}{3}\cos\pi = -\sqrt{3}$

POPU

At $(2\pi/3, 2\pi/3)$

$r = 2\sin\frac{2\pi}{3}\cos 2\pi = \sqrt{3}$

$s = \sin\frac{8\pi}{3} = \frac{\sqrt{3}}{2}$, $t = 2\sin\frac{2\pi}{3}\cos 2\pi = \sqrt{3}$

At pt $(\pi/3, \pi/3)$

$rt - s^2 = 3 - \left(-\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$

$r < 0$ maxima at pt $(\pi/3, \pi/3)$

At pt $(2\pi/3, 2\pi/3)$

$rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$ $\cancel{r>0}$ $r > 0$

then pt $(2\pi/3, 2\pi/3)$ minima

max. value $f(\pi/3, \pi/3) = \sin\frac{\pi}{3} . \sin\frac{\pi}{3} \sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{8}$

min. value $f\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} \sin\frac{2\pi}{3} \sin\frac{4\pi}{3} = -\frac{3\sqrt{3}}{8}$

Imp

Q.4 In a ΔABC, find the maximum value of $\cos A \cos B \cos C$

Imp

Q.5 Find the pt where the function $x^3y^2(1-x-y)$ has maximum or minimum value and also find the value of function at these point.

Q.6 Divide 24 into three parts such that continued product of first, square of second and cube of third is a maximum (RTU: 2008)

Ans-4 $\because A + B + C = 180°$ $\Rightarrow$ $C = 180° - (A+B)$

$f(A,B) = \cos A \cos B \cos(180° - (A+B)) = -\cos A \cos B \cos(A+B)$

$$\frac{\partial f}{\partial A} = -\cos B\left[\cos(A+B)\sin A - \cos A\sin(A+B)\right]$$

$$\frac{\partial f}{\partial A} = +\cos B\sin(A+A+B) = \cos B\sin(2A+B) \quad — ①$$

$$\frac{\partial f}{\partial B} = -\cos A\left[-\cos(A+B)\sin B - \cos B\sin(A+B)\right]$$

$$\frac{\partial f}{\partial B} = \cos A\sin(A+B+B) = \cos A\sin(A+2B) \quad — ②$$

~~for~~ maxima :-

$$\frac{\partial f}{\partial A} = 0 \quad , \quad \frac{\partial f}{\partial B} = 0$$

$\cos B\sin(2A+B) = 0$, $\cos A\sin(A+2B) = 0$

$B = \frac{\pi}{2}$, $2A+B = 0, \pi, 2\pi$; $A = \frac{\pi}{2}$, $A+2B = 0, \pi, 2\pi$

$$\begin{aligned} 2A + B &= \pi \\ 2A + 4B &= 2\pi \\ \hline B = \frac{\pi}{3}, &\ A = \frac{\pi}{3} \end{aligned} \qquad , \qquad \begin{aligned} 2A + B &= 2\pi \\ 2A + 4B &= 4\pi \\ \hline B = \frac{2\pi}{3}, &\ A = \frac{2\pi}{3} \end{aligned}$$

$$r = \frac{\partial^2 f}{\partial A^2} = 2\cos B\cos(2A+B) \qquad t = \frac{\partial^2 f}{\partial B^2} = 2\cos A\cos(A+2B)$$

$$\frac{\partial^2 f}{\partial A\partial B} = \cos A\cos(A+2B) + \sin(A+2B)(-\sin A)$$

$$s = \frac{\partial^2 f}{\partial A\partial B} = \cos(A+A+2B) = \cos(2A+2B)$$

At $(0,0)$

$r = \frac{\partial^2 f}{\partial A^2} = 2$, $t = \frac{\partial^2 f}{\partial B^2} = 2$, $s = 1$

$rt - s^2$, $r > 0$

$4 - 1 = 3 > 0$ minima

POPU

Page No.
Date:

At $(\pi/3, \pi/3)$

$r = 2\left(\frac{1}{2}\right)(-1) = -1$, $s = \left(\frac{-1}{2}\right)$, $t = 2\left(\frac{1}{2}\right)(-1) = -1$

$rt - s^2$

$(-1)(-1) - \left(\frac{-1}{2}\right)^2 = 1 - \frac{1}{2} = \frac{1}{2} > 0$

$r < 0$ then f is maxima

max. value $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = -\cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{2\pi}{3} = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right) = \frac{1}{8}$

At $(2\pi/3, 2\pi/3)$

$r = 2\left(\frac{-1}{2}\right)(-1) = -1$, $s = \cos\frac{8\pi}{3} = \left(\frac{-1}{2}\right)$

$t = 2\left(\frac{-1}{2}\right)(1) = -1$

$rt - s^2 = (-1)(-1) - \left(\frac{-1}{2}\right)^2 = 1/2 > 0$

$r < 0$ then f is maxima

max. value $f\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = -\cos\frac{2\pi}{3}\cos\frac{2\pi}{3}\cos\frac{4\pi}{3}$

$= -\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right) = \frac{1}{8}$ Ay

Ay 5 $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = x^2y^2(3 - 4x - 3y)$

$\frac{\partial f}{\partial y} = 2yx^3 - 2yx^4 - 3y^2x^3 = x^3y(2 - 2x - 3y)$

For maxima and minima

$\frac{\partial f}{\partial x} = x^2y^2(3-4x-3y) = 0$, $\frac{\partial f}{\partial y} = x^3y(2-2x-3y) = 0$

$x=0, y=0, 3 = 4x+3y$, $x=0, y=0$, $2 = 2x+3y$

$\cancel{(0, 2/3)}$ $\cancel{(1, 0)}$ $\cancel{(0,1)}$ $\cancel{(1,0)}$ $(0,1)$ $(1,0)$

$(3/4, 0)$

$(0, 2/3)$

$\cancel{2x + 2y = 2}$

$\cancel{2x + 3y = 2}$

$y = 0$ $\cancel{\boxed{x=1}}$

$3 = 4x + 3y$

$4 = 4x + 6y$

$1 = 4y \Rightarrow y = \frac{1}{4}, x = 5/8$

$\frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3 = r$

$\frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2 = s$

$\frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6yx^3 = t$

At pt $(1, 0)$

$r = 0$, $s = 0$, $t = 0$

At pt $(3/4, 0)$

$r = 0$, $s = 0$, $t = 2 \times \frac{27}{4 \times 4 \times 4} - 2 \times \frac{27 \times 3}{4 \times 4^3} = \frac{2 \times 27}{2 \times 4 \times 4 \times 4}\left(\frac{1}{4}\right)$

$t = \frac{27}{128}$

$rt - s^2 = 0$

At pt $\cancel{(\frac{1}{4}, 5/8)}$ $(5/8, \frac{1}{4})$

$r = \cancel{6\left(\frac{1}{4}\right)\left(\frac{25}{64}\right)}$ $6xy^2(1 - 2x - y)$

$r = 6\left(\frac{5}{8}\right)\left(\frac{1}{4} \times \frac{1}{4}\right)\left(1 - 2\left(\frac{5}{8}\right) - \frac{1}{4}\right)$

$\frac{3}{\cancel{6}} \times \frac{5}{\cancel{8}_4} \times \frac{1}{16}\left(\frac{4 - 5 - 1}{4}\right) = \frac{-15}{128}$

$s = x^2y(6 - 8x - 9y)$

$s = \frac{5}{8} \times \frac{5}{8} \times \frac{1}{4}\left(6 - 8 \times \frac{5}{8} - \frac{9}{4}\right)$

$s = \frac{25}{64} \times \frac{1}{4}\left[\frac{24 - 20 - 9}{4}\right] = \frac{-125}{1024}$

$t = \frac{2 \times 125}{8 \times 64} \quad 2x^3(1 - x - 3y)$

$t = \frac{2 \times 5 \times 5 \times 5}{64 \times 8}\left(1 - \frac{5}{8} - \frac{3}{4}\right) = \frac{125}{64 \times 4}\left(\frac{8-5-6}{8}\right)$

$t = -\frac{125 \times 3}{2048} = \frac{-375}{2048}$

$rt - s^2$

$\frac{-15}{128} \times \frac{-375}{2048} - \left(\frac{-125}{1024}\right)^2$

$\frac{5625}{1046976} - \frac{15625}{261144} < 0$

Q. 3 $x + y + z = 24$

$x \cdot y^2 \cdot z^3 = f(x, y, z)$

$f(y, z) = (24 - y - z)\, y^2 z^3$

$f(y, z) = 24y^2z^3 - y^3z^3 - y^2z^4$

$\frac{\partial f}{\partial y} = 48yz^3 - 3y^2z^3 - 2yz^4 = yz^3(48 - 3y - 2z) = 0$

$3y + 2z = 48 \quad (1)$

$\frac{\partial f}{\partial z} = 72y^2z^2 - 3y^3z^2 - 4y^2z^3 = y^2z^2(72 - 3y - 4z) = 0$

$3y + 4z = 72 \quad (2)$

$3y + 2z = 48$

$3y + 4z = 72$

$-2z = -24$

$\boxed{z = 12} \quad \boxed{y = 8}$

$\frac{\partial^2 f}{\partial y^2} = 48z^3 - 6yz^3 - 2z^4 = z^3(48 - 6y - 2z) = r$

$\frac{\partial^2 f}{\partial z^2} = 144y^2z - 6y^3z - 12y^2z^2 = y^2z(144 - 6y - 12z) = t$

$\frac{\partial^2 f}{\partial z \partial y} = 144yz^2 - 9y^2z^2 - 8yz^3 = yz^2(144 - 9y - 8z) = s$

At pt (~~[illegible]~~ (8,12)

$r = (12)^3 (48 - 6(8) - 2(12))$

$r = 144 \times 12 (48 - \cancel{48} - 24) = -ve = -41472$

$s = (8)(12)^2 (144 - 9\times8 - 8\times12)$

$s = 8\times144 (144 - 72 - 96) = -ve = -27648$

$t = (8)^2 \times 12 (144 - 6\times8 - 12\times12) = -ve = -36864$

$rt - s^2 = (-ve)(-ve) - (-ve)^2$

$+ve - (ve)^2$

$rt - s^2 = (-41472)(-36864) - (27648)^2$

$rt - s^2 > 0$

168
144
-24
= 144

$r < 0$

1528823808
-764411904
764411904

768
-48

maxima

$f(y,z) = f(8,12) = (24 - 8 - 12)(8)^2 (12)^3$

$(4) \times 64 \times 144 \times 12$

$= 442368$

POPU

Page No.
Date:

Imp

Lagrange's Multipliers Method

$\Rightarrow$ Let $f(x,y,z)$ be the function of three variables and the variables be connected by relation $\phi(x,y,z) = 0$

then we have Lagrangion function

$$F(x,y,z,\lambda) = f(x,y,z) + \lambda\,\phi(x,y,z) \;;\quad \lambda \neq 0 \quad \text{—(1)}$$

where λ is called lagrangian multipliers.

Now for extreme (Minima or maxima)

$$\frac{\partial F}{\partial x} = 0 \text{ —(2)}, \quad \frac{\partial F}{\partial y} = 0 \text{ —(3)}, \quad \frac{\partial F}{\partial z} = 0 \text{ —(4)}$$

$$\frac{\partial F}{\partial \lambda} = 0 \text{ —(6)}$$

These equations are called Lagrangian equations.

Q. Find the point upon the plane $ax+by+cz=p$ at which the function $f(x,y,z) = x^2+y^2+z^2$ has a minimum value and find minimum value of F.

Ans - We have $f(x,y,z) = x^2+y^2+z^2$ —(1)

function f is connected by the relation

$\phi(x,y,z) : ax+by+cz-p = 0$ —(2)

Then lagrangian's Multipliers method

$F = f + \lambda\phi$, where λ is Lagrangian's multipliers

$F = (x^2+y^2+z^2) + \lambda(ax+by+cz-p)$ —(3)

For extreme value

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda a = 0$ —(4)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda b = 0$ —(5)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda c = 0$ —(6)

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow ax+by+cz-p = 0$ —(7)

multiplying (4), (5), & (6) by x, y & z respectively and adding, we have

$(2x^2+2y^2+2z^2)+\lambda(ax+by+cz)=0$

$2(x^2+y^2+z^2)+\lambda(ax+by+cz)=0$

using eqn -(1) & (2)

$2f+\lambda P$

$\lambda=\frac{-2f}{P}$

from eqn -(4)

$2x+\left(\frac{-2f}{P}\right)a=0$

$x=\frac{af}{P}$

from (5) $2y+\left(\frac{-2f}{P}\right)b=0 \Rightarrow y=\frac{bf}{P}$

from (6) $2z+\left(\frac{-2f}{P}\right)c=0 \Rightarrow z=\frac{cf}{P}$

from eqn -(7)

$a\left(\frac{af}{P}\right)+b\left(\frac{bf}{P}\right)+c\left(\frac{cf}{P}\right)=P$

$a^2f+b^2f+c^2f=P^2$

$f=\frac{P^2}{a^2+b^2+c^2}$ which is required min. value of given function.

Q.2 If $u=ax^2+by^2+cz^2$ where $x^2+y^2+z^2=1$ and $lx+my+nz=0$ Prove that extreme Values of u satified the equation

$\frac{l^2}{a-u}+\frac{m^2}{b-u}+\frac{c^2}{c-u}=0$ also interpret the result geometrically

Ans- We have $U = ax^2+by^2+cz^2$ —— (1)

and $\phi_1 = (x^2+y^2+z^2-1) = 0$ —— (2)

$\phi_2 = (lx+my+nz) = 0$ —— (3)

Lagrange's Multiplier equation :-

$F = U + \lambda\phi_1 + \mu\phi_2$ where λ & μ are langrange's multiplier

$F = (ax^2+by^2+cz^2) + \lambda(x^2+y^2+z^2-1) + \mu(lx+my+nz)$ —— (4)

Now

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2ax + 2\lambda x + \mu l = 0$ —— (5)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2by + 2\lambda y + \mu m = 0$ —— (6)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2cz + 2\lambda z + \mu n = 0$ —— (7)

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x^2+y^2+z^2-1 = 0 \Rightarrow x^2+y^2+z^2 = 1$ —— (8)

$\frac{\partial F}{\partial \mu} = 0 \Rightarrow lx+my+nz = 0$ —— (9)

Multiplying (5), (6) & (7) by x, y & z respectivelly, when

$$2ax^2 + 2\lambda x^2 + \mu l x = 0$$
$$2by^2 + 2\lambda y^2 + \mu m y = 0$$
$$2cz^2 + 2\lambda z^2 + \mu n z = 0$$

On adding :-

$$2(ax^2+by^2+cz^2) + 2\lambda(x^2+y^2+z^2) + \mu(lx+my+nz) = 0$$

using eq^n - (1), (8) & (9)

$$2U + 2\lambda(1) + 0 = 0$$

$$\boxed{\lambda = -U}$$

putting λ in eq^n 5, 6 & 7

$2ax - 2ux + \lambda l = 0 \Rightarrow (2a - 2u)x = -\lambda l$
$2by - 2uy + \lambda m = 0 \Rightarrow 2y(b-u) = -\lambda m$
$2cz - 2uz + \lambda n = 0 \Rightarrow 2z(c-u) = -\lambda n$

$$x = \frac{-\lambda l}{2(a-u)}, \quad y = \frac{-\lambda m}{2(b-u)}, \quad z = \frac{-\lambda n}{2(c-u)}$$

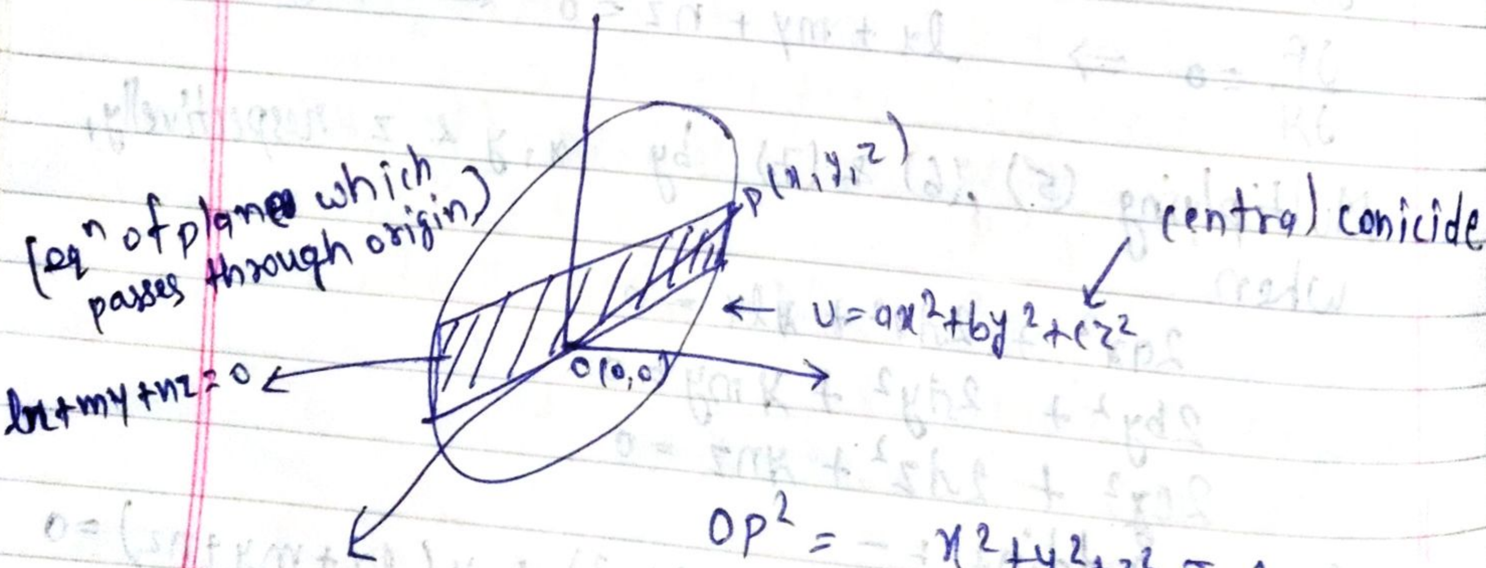
from eq^n 3 $lx + my + nz = 0$

$$l\left(\frac{\lambda l}{2(u-a)}\right) + m\left(\frac{\lambda m}{2(u-b)}\right) + n\left(\frac{\lambda n}{2(u-c)}\right)$$

$$\frac{\lambda}{2}\left(\frac{l^2}{u-a} + \frac{m^2}{u-b} + \frac{n^2}{u-c}\right) = 0$$

$\lambda \neq 0$

$$\Rightarrow \frac{l^2}{u-a} + \frac{m^2}{u-b} + \frac{n^2}{u-c} = 0$$



$$OP^2 = x^2+y^2+z^2 = 1$$

Q. In a tringle ΔABC find that maxima and minima of
$U = \sin A \sin B \sin C$

Ans- In ΔABC $A+B+C=\pi$, $U = \sin A \sin B \sin C$ — ①

$\phi = A+B+C-\pi = 0$ — ②

Lag rangion function is
$F(A,B,C) = U + \lambda\phi$

$F = \sin A \sin B \sin C + \lambda(A+B+C-\pi)$ — ③

$\frac{\partial F}{\partial A} = 0 \Rightarrow \cos A \sin B \sin C + \lambda = 0$ — ④

$\frac{\partial F}{\partial B} = 0 \Rightarrow \cos B \sin A \sin C + \lambda = 0$ — ⑤

$\frac{\partial F}{\partial C} = 0 \Rightarrow \cos C \sin A \sin B + \lambda = 0$ — ⑥

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow A+B+C-\pi = 0$

$A+B+C = \pi$ — ⑦

from ④ & ⑤

$\cos A \sin B \sin C = \cos B \sin A \sin C$

$\sin C(\cos A \sin B - \cos B \sin A) = 0$

$\sin C = 0$ $\quad\quad \sin A \cos B - \cos A \sin B = 0$

$\boxed{C=0}$ $\quad\quad \sin(A-B) = 0$

$\boxed{A=B}$

Similar from ⑤ & ⑥ $\boxed{B=C}$

Similar from ④ & ⑥ $\boxed{A=C}$

$\boxed{A=B=C}$ — ⑧

from eqn - ⑦ $A+B+C=\pi$

$3A = \pi \Rightarrow \boxed{A = \pi/3}$

$$\boxed{A=B=C=\frac{\pi}{3}}$$

$U = \sin A \sin B \sin C$

$U = \sin\frac{\pi}{3} \sin\frac{\pi}{3} \sin\frac{\pi}{3} = \frac{3\sqrt{3}}{8}$ which shows maxima or minima of given function.

Q. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}$ where

$x = \xi \cos\alpha - \eta \sin\alpha$

$y = \xi \sin\alpha + \eta \cos\alpha$

Ans - $U = f(x, y)$

$x = \phi(\xi, \eta)$, $y = \psi(\xi, \eta)$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$= \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial \xi}(\xi \cos\alpha - \eta \sin\alpha) + \frac{\partial u}{\partial y} \cdot \frac{\partial}{\partial \xi}(\xi \sin\alpha + \eta \cos\alpha)$$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y} \cdot \sin\alpha \quad \text{—} \ (1)$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta}$$

$$= \frac{\partial u}{\partial x}\frac{\partial}{\partial \eta}(\xi \cos\alpha - \eta \sin\alpha) + \frac{\partial u}{\partial y}\frac{\partial}{\partial \eta}(\xi \sin\alpha + \eta \cos\alpha)$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}(-\sin\alpha) + \frac{\partial u}{\partial y}(\cos\alpha) \quad \text{—} \ (2)$$

from eqn — (1)

$$\frac{\partial u}{\partial \xi} = \cos\alpha \frac{\partial u}{\partial x} + \sin\alpha \frac{\partial u}{\partial y}$$

POPU

$$\frac{\partial}{\partial \xi} = \cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y} \quad \text{--- (3)}$$

Again from (1)

$$\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left(\cos\alpha \frac{\partial u}{\partial x} + \sin\alpha \frac{\partial u}{\partial y}\right)$$

using (1)

$$\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right) = \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y}\right)\left(\cos\alpha \frac{\partial u}{\partial x} + \sin\alpha \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial \xi^2} = \cos^2\alpha \frac{\partial^2 u}{\partial x^2} + \sin\alpha\cos\alpha \frac{\partial^2 u}{\partial x \partial y} + \sin\alpha\cos\alpha \frac{\partial^2 u}{\partial y \partial x} + \sin^2\alpha \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial \xi^2} = \cos^2\alpha \frac{\partial^2 u}{\partial x^2} + \sin^2\alpha \frac{\partial^2 u}{\partial y^2} + 2\sin\alpha\cos\alpha \frac{\partial^2 u}{\partial x \partial y} \quad \text{--- (4)}$$

from eq^n --- (2)

$$\frac{\partial u}{\partial \eta} = -\sin\alpha \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}(\cos\alpha)$$

$$\frac{\partial}{\partial \eta} = -\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y} \quad \text{--- (4)}$$

Again eq^n --- (2)

$$\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right) = \frac{\partial}{\partial \eta}\left(-\sin\alpha \frac{\partial u}{\partial x} + \cos\alpha \frac{\partial u}{\partial y}\right)$$

using (4)

$$\frac{\partial^2 u}{\partial \eta^2} = \left(-\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y}\right)\left(-\sin\alpha \frac{\partial u}{\partial x} + \cos\alpha \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial \eta^2} = \sin^2\alpha \frac{\partial^2 u}{\partial x^2} - 2\sin\alpha\cos\alpha \frac{\partial^2 u}{\partial x \partial y} + \cos^2\alpha \frac{\partial^2 u}{\partial y^2} \quad (5)$$

$$\left(\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\right)$$

Page No. :
Date :

eq^n — ④ + ⑤

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = (\sin^2\alpha + \cos^2\alpha)\frac{\partial^2 u}{\partial x^2} + \cancel{2\sin\alpha\cos\alpha\frac{\partial^2 u}{\partial x \partial y}}$$

$$- \cancel{2\sin\alpha\cos\alpha\frac{\partial^2 u}{\partial x \partial y}} + \cos^2\alpha\frac{\partial^2 u}{\partial y^2}$$

$$+ \sin^2\alpha \frac{\partial^2 u}{\partial y^2}$$

$$\boxed{\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}$$

Q. If $u = f(r)$, where $r^2 = x^2 + y^2$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$

Ans –

$$r^2 = x^2 + y^2$$

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2)$$

$$2r \cdot \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$u = f(r)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(f(r))$$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} = f'(r) \cdot \frac{x}{r}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{f'(r)}{r} \cdot x\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} + x \; \frac{r f''(r)\frac{\partial r}{\partial x} - f'(r)\frac{\partial r}{\partial x}}{r^2}$$

Date : / /

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} + \frac{x}{\cancel{r}} \cdot \frac{r \times \frac{x}{r} f''(r) - f'(r)\frac{x}{r}}{r^2}$$

$$= \frac{f'(r)}{r} + \frac{x^2}{r^2}\left(f''(r) - \frac{1}{r} f'(r) \right) \quad \text{—(1)}$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} + \frac{y^2}{r^2}\left(f''(r) - \frac{1}{r} f'(r) \right) \quad \text{—(2)}$$

$$\text{eq}^n \text{ — (1) + (2)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} + \frac{f'(r)}{r} + \frac{x^2+y^2}{r^2}\left(f''(r) - \frac{1}{r} f'(r) \right)$$

$$= \frac{2f'(r)}{r} + \frac{\cancel{r^2}}{\cancel{r^2}}\left(f''(r) - \frac{1}{r} f'(r) \right)$$

$$= f''(r) + \frac{2f'(r) - f'(r)}{r}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)} \quad \text{(H.P.)}$$

chapter - 1

Beta and Gamma Function :-

Beta Function :- The Beta function denoted by $B(m,n)$, is a proper definite integral defined as.

$$\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,.dx \qquad m,n>0}$$

Properties & results of Beta function :-

1.) $B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,.dy$

2.) $B(m,n) = B(n,m) \longrightarrow$ Symmetry property

3.) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \,.\, \cos^{2n-1}\theta \,.d\theta \longrightarrow$ Trigonometric representation of $B(m,n)$

Solution

(i) $B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,.dy$

By definition of Beta function

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,.dx$ — (1)

let $x = \frac{1}{1+y}$ $\Rightarrow$ $1+y = \frac{1}{x} \Rightarrow y = \frac{1}{x}-1$

$dx = \frac{-1}{(1+y)^2}\,.dy$

$x=0, \quad y=\infty$

$x=1, \quad y=0$

$$B(m,n) = \int_\infty^0 \left(\frac{1}{y+1}\right)^{m-1}\left(1-\frac{1}{y+1}\right)^{n-1}\cdot\left(\frac{-1}{(1+y)^2}\right)\cdot dy$$

$$= -\int_\infty^0 \left(\frac{1}{y+1}\right)^{m-1}\left(\frac{\cancel{1}+y-\cancel{1}}{y+1}\right)^{n-1}\left(\frac{1}{(1+y)^2}\right)\cdot dy$$

$$= \int_0^\infty \frac{(y)^{n-1}}{(y+1)^{m-1+n-1+2}} \cdot dy$$

$$B(m,n) = \int_0^\infty \frac{y^{n-1}}{(y+1)^{m+n}} \cdot dy$$

(iii) $B(m,n) = 2\int^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \cdot d\theta$

By definition of Beta function

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx$$

put $x = \sin^2\theta$
$dx = 2\sin\theta\cos\theta \cdot d\theta$

$x=1$ then $\theta = \sin^{-1}(1) = \pi/2$

$x=0$ then $\theta = \sin^{-1}(0) = 0$

$$\therefore \quad B(m,n) = \int_0^{\pi/2} \sin^{2(m-1)}\theta \, (1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta \cdot d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin\theta^{2m-2+1} (1-\sin^2\theta)^{n-1} \cos\theta \cdot d\theta$$

$$= 2\int_0^{\pi/2} \sin\theta^{2m-1} \cos\theta^{2n-2+1} \cdot d\theta$$

$$\boxed{B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta}$$

Gamma Function :- The Gama function, denoted by $\sqrt{(n)}$, is an improper integral defined as

$$\boxed{\sqrt{n} = \int_0^\infty e^{-x} x^{n-1} dx} \; ; \; n > 0$$

Properties of Gamma function :—

(i) $\Gamma(n+1) = n\Gamma n = \lfloor n = n!$

Imp (ii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Relation b/w Beta and Gamma function

Imp

$$\boxed{B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

Imp

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \cdot d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

Q. $\displaystyle\int_0^\infty \frac{x^2(1+x^4)}{(1+x)^{10}} \cdot dx$

Ans- ~~We know that~~ we have

$$\int_0^\infty \frac{x^2(1+x^4)}{(1+x)^{10}} \cdot dx$$

we know that $B(m,n) = \displaystyle\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} \cdot dy$

$$\int_0^\infty \frac{x^2}{(1+x)^{10}} \cdot dx + \int_0^\infty \frac{x^6}{(1+x)^{10}} \cdot dx$$

$$\int_0^\infty \frac{x^{3-1}}{(1+x)^{7+3}} \cdot dx + \int_0^\infty \frac{x^{7-1}}{(1+x)^{3+7}} \cdot dx$$

$B(7,3) + B(3,7)$ $\quad (\because B(m,n) = B(n,m))$

$B(7,3) + B(7,3)$

$= 2B(7,3)$ $\quad \because \left(B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right)$

$$= \frac{\Gamma 7 \, \Gamma 3}{\Gamma 10} = \frac{\lfloor 6 \; \lfloor 2}{\lfloor 9}$$

$$2 \times \frac{\cancel{6} \times 2}{9 \times 8 \times 7 \cancel{6}} = \frac{1}{9 \times 7 \times 2} = \frac{1}{126} \text{ Ans}$$

Q. Show that

$$B(m,n) = B(m+1,n) + B(m,n+1)$$

Ans- RHS

$$B(m+1,n) + B(m,n+1) \quad \left[\because B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}\right]$$

$$\frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+1+n)} + \frac{\Gamma m \; \Gamma(n+1)}{\Gamma(m+n+1)} \quad \left[\because \Gamma(n+1) = n\Gamma n\right]$$

$$\frac{m\Gamma m \, \Gamma n}{(m+n)\Gamma(m+n)} + \frac{\Gamma m \cdot n\Gamma n}{(m+n)\Gamma(m+n)}$$

$$\frac{\Gamma m \, \Gamma n}{\cancel{(m+n)}\,\Gamma m+n}\cancel{(m+n)} = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$= B(m,n)$$

(H.P.)

eg- $I = \int_0^{\pi/6} \cos^4 3\theta \; \sin^2 6\theta \cdot d\theta$

Ans- $I = \int_0^{\pi/6} \cos^4 3\theta \; (2\sin 3\theta \cos 3\theta)^2 \cdot d\theta$

$$I = 4\int_0^{\pi/6} \cos^6 3\theta \; \sin^2 3\theta \cdot d\theta$$

let $3\theta = t$ | $3\theta = 0,\; t = 0$

$\cancel{\theta = \frac{t}{3}}$ | $3\theta = \frac{\pi}{6}$ $\quad t = \frac{\pi}{2}$

$d\theta = \frac{dt}{3}$

$I = \frac{4}{3}\int_0^{\pi/2} \cos^6 t \sin^2 t \, dt$

$\left[\because \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2} \, \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}} \right]$

$I = \frac{4}{3}\left(\frac{\Gamma\frac{2+1}{2} \, \Gamma\frac{6+1}{2}}{2\,\Gamma\frac{2+6+2}{2}} \right)$

~~$I = \frac{4}{3} \frac{\frac{\Gamma 3}{\Gamma 2} \frac{\Gamma 7}{\Gamma 2}}{2 \Gamma 5}$~~ $=$ ~~$\frac{2\cancel{4}}{3} \times \frac{\cancel{6} \cancel{L4} \; L12}{\cancel{2} \times \cancel{L4}}$~~

~~$I = \frac{2}{3} \times$~~ $\quad I = \frac{\cancel{4}2}{3}\left(\frac{\Gamma\frac{3}{2} \, \Gamma\frac{7}{2}}{\cancel{2}\,\Gamma 5} \right)$

$I = \frac{\frac{\cancel{2}}{\cancel{3}} \times \frac{1}{\cancel{2}}\sqrt{\pi} \times \frac{5}{2} \times \frac{\cancel{3}}{2} \times \frac{1}{2}\sqrt{\pi}}{4 \times 3 \times 2 \times 1} = \frac{\pi \times 5}{192} = \frac{5\pi}{192}$

Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof:- By defination of Gamma function:-

$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx \quad , \quad n > 0 \quad \text{---} (1)$

put $n = \frac{1}{2}$

$\Gamma\frac{1}{2} = \int_0^\infty e^{-x} x^{\frac{1}{2}-1} \, dx = \int_0^\infty e^{-x} x^{-1/2} \, dx$

let $x = u^2$ $\quad | \quad x = 0, \; u = 0$

$dx = 2u \, du$ $\quad | \quad x = \infty, \; u = \infty$

$\Gamma\frac{1}{2} = \int_0^\infty e^{-u^2} (u^2)^{-1/2} \cdot 2u\,du$

POPU

Page No.

Date

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-u^2} \cdot du \quad \text{—— (2)}$$

Similarly

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-v^2} \cdot dv \quad \text{— (3)}$$

eq^n – (2) × (3)

$$\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = 4\int_0^\infty \int_0^\infty e^{-u^2} \cdot e^{-v^2} \cdot du\,dv$$

$$\left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} \cdot du\,dv \quad \text{— (4)}$$

polar form

putting $v = r\sin\theta$

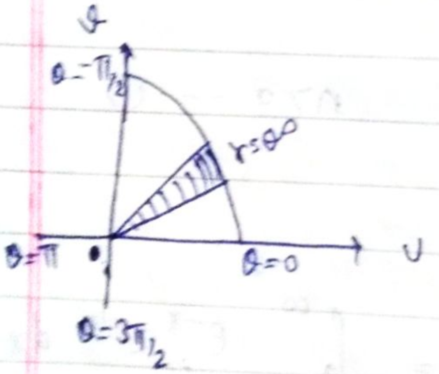
$u = r\cos\theta$

$$u^2+v^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(1)$$

$$\boxed{r^2 = u^2+v^2}$$

also $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{v}{u}$

$$\theta = \tan^{-1}\left(\frac{v}{u}\right)$$



$u=0,\ v=0$

$u=\infty,\ v=\infty$

$r^2 = u^2+v^2$

$u=0,\ v=0 \quad , r=0$

$u=\infty,\ v=\infty \quad , r=\infty$

$$\theta = \tan^{-1}\left(\frac{v}{u}\right)$$

if $v=0$, $\theta = \tan^{-1}(0) = 0$

if $u=0$, $\theta = \tan^{-1}(\infty) = \pi/2$

and $\boxed{du\,dv = r\,d\theta\,dr}$

$$\boxed{\begin{array}{l} \theta = 0 \text{ to } \theta = \pi/2 \\ r = 0 \text{ to } r = \infty \end{array}}$$

in eq^n - (4)

$$\left(\sqrt{\frac{1}{2}}\right)^2 = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\cdot r\,d\theta\,dr$$

let $r^2 = t$

$2r\,dr = dt$

$$\left(\sqrt{\frac{1}{2}}\right)^2 = \frac{4}{2}\int_0^{\pi/2}\left(\int_0^{\infty} e^{-t}\,dt\right)d\theta$$

$$= 2\int_0^{\pi/2}\left[-e^{-t}\right]_0^{\infty}\cdot d\theta$$

$$= -2\int_0^{\pi/2}\left[e^{-\infty} - e^{0}\right]\cdot d\theta$$

$$= -2\int_0^{\pi/2}\left[0-1\right]\cdot d\theta = 2\int_0^{\pi/2} d\theta$$

$$\left(\sqrt{\frac{1}{2}}\right)^2 = \cancel{2}\times\frac{\pi}{\cancel{2}} \Rightarrow \boxed{\sqrt{\frac{1}{2}} = \sqrt{\pi}}$$

Important Result

Imp

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Q. Show that $\int_0^{\infty}\frac{1}{1+y^4}\,dy = \frac{\pi}{2\sqrt{2}}$

Ay- let $y^2 = \tan\theta$

$2y\,dy = \sec^2\theta\cdot d\theta$

$dy = \frac{\sec^2\theta}{2\sqrt{\tan\theta}}\cdot d\theta$

$y = 0, \theta = 0$

$y = \infty, \theta = \pi/2$

$$\int_0^{\pi/2} \frac{1}{1+\cancel{\tan^2\theta}} \times \frac{\sec^2\theta}{2\sqrt{\tan\theta}} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{\cot\theta} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^{1/2}\theta \, \sin^{-1/2}\theta \cdot d\theta$$

$$\left[\because \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}} \right]$$

$$= \frac{1}{2} \frac{\Gamma\frac{-1/2+1}{2}\,\Gamma\frac{\frac{1}{2}+1}{2}}{2\,\Gamma\frac{\frac{1}{2}-\frac{1}{2}+2}{2}}$$

$$= \frac{1}{2} \times \cancel{2} \times \frac{1}{2} \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{\Gamma 1} \qquad \left(\because \Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi} \right)$$

$$= \frac{1}{4}\,\Gamma\frac{1}{4}\,\Gamma 1-\frac{1}{4}$$

$$= \frac{1}{4} \times \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}\,\pi}{4} = \frac{\pi}{2\sqrt{2}}$$

$$\boxed{\int_0^{\infty} \frac{1}{1+y^4} \cdot dy = \frac{\pi}{2\sqrt{2}}} \quad (\text{H.P.})$$

Relation b/w Beta and Gamma function:-

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$$\Gamma m = \int_0^{\infty} e^{-x} x^{m-1} \cdot dx \quad \text{—— (1)}$$

$$\Gamma n = \int_0^\infty e^{-y}\, y^{n-1} \cdot dy \quad \text{——— (2)}$$

~~eqⁿ — (1) × (2)~~

$$\cancel{\Gamma m\, \Gamma n = \int_0^\infty \int_0^\infty e^{-x} \cdot e^{-y} \cdot x^{m+n-2} \cdot dx\, dy}$$

let $x = u^2$, $y = v^2$

$\cancel{2x}\, dx = 2u\, du$ $\quad dy = 2v\, dv$

$$\Gamma m = \int_0^\infty e^{-u^2}\, u^{2m-2} \cdot (2u)\, du$$

$$\Gamma m = 2\int_0^\infty e^{-u^2}\, u^{2m-1} \cdot du \quad \text{—— (3)}$$

Similarly $\quad \Gamma n = 2\int_0^\infty e^{-v^2} \cdot v^{2n-1} \cdot dv$ —— (4)

eqⁿ — (3) × (4)

$$\Gamma m\ \Gamma n = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}\, u^{2m-1} \cdot v^{2n-1}\ du\, dv$$

let $u = r\cos\theta$ $\quad r = 0$ to ∞

$v = r\sin\theta$ $\quad \theta = 0$ to $\pi/2$

$du\,dv = r\,d\theta\,dr$

$$\Gamma m\ \Gamma n = 4\int_{\theta=0}^{\pi/2} \int_{r=0}^\infty e^{-(r^2(\cos^2\theta+\sin^2\theta))}\ (r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1} \cdot r\,d\theta\,dr$$

$$\Gamma m\,\Gamma n = 4\int_{\theta=0}^{\pi/2}\int_0^\infty e^{-r^2}\ \cos^{2m-1}\theta\ \sin^{2n-1}\theta \cdot r^{2m+2n-2+1} \cdot d\theta\, dr$$

$$\Gamma m\,\Gamma n = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\ \cos^{2m-1}\theta\ \sin^{2n-1}\theta \cdot r^{2m+2n-2} \cdot r \cdot d\theta\, dr$$

let $r^2 = t$ $\quad 2r\,dr = dt$

$$\Gamma m\,\Gamma n = 2\int_0^{\pi/2}\int_0^\infty e^{-t}\ \cos^{2m-1}\theta\ \sin^{2n-1}\theta\ \cancel{r^{2(m+n-1)}}\ t^{m+n-1} \cdot d\theta \cdot dt$$

$$\Gamma m \, \Gamma n = \left[2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \cdot d\theta \right]$$

$$\left[2\int_0^{\infty} e^{-r^2} \cdot r^{2(m+n)-1} \cdot dr \right]$$

$$\Gamma m \, \Gamma n = B(m,n) \, I_1$$

$$\left(\because \Gamma n = \int_0^{\infty} e^{-x} \, x^{n-1} \cdot dx \right)$$

$$I_1 = \int_0^{\infty} e^{-t} \, t^{(m+n-1)} \cdot dt$$

$$I_1 = \Gamma m+n$$

$$\Gamma m \, \Gamma n = B(m,n) \, \Gamma m+n$$

$$\boxed{B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$ (H.P.)

Prove $\int_0^{\pi/2} \sin^m\theta \, \cos^n\theta \cdot d\theta = \dfrac{\Gamma \frac{m+1}{2} \, \Gamma \frac{m+1}{2}}{2\Gamma \frac{m+n+2}{2}}$

Solve:- We have already proved that

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \cdot d\theta$$

$$\left(\because B(m+n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \right.$$

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, .d\theta$$

POPU

Page No.
Date

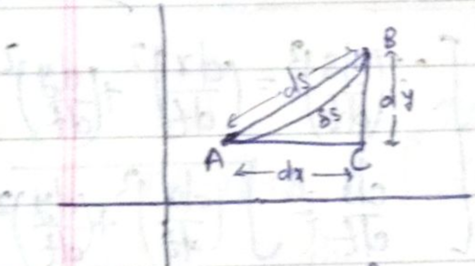
$$m \to \frac{m+1}{2} \quad , \quad n \to \frac{n+1}{2}$$

$$\frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{m+n+2}{2}} = 2\int_0^{\pi/2} \sin^{2(\frac{m+1}{2})-1}\theta \, \cos^{2(\frac{n+1}{2})-1}\theta \cdot d\theta$$

$$\frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}} = \int_0^{\pi/2} \sin^m\theta \, \cos^n\theta \cdot d\theta$$

(M.P.)

Surfaces and volumes of Revalution of solids ⇒

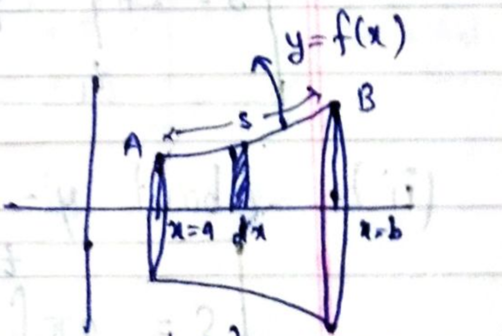


s - length of arc

$$\boxed{(ds)^2 = (dx)^2 + (dy)^2}$$

Surfaces ~~Volume~~ of revalution of solids :—

let $y=f(x)$ be a curve



(i) If AB revolving about x-axis:—

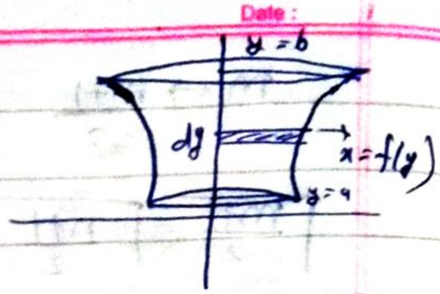
$$S = 2\pi \int y \cdot ds$$

$$S = 2\pi \int_{x=a}^{x=b} y \cdot \frac{ds}{dx} \cdot dx$$

$$\left[\because \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \qquad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \right]$$

$$\boxed{S = 2\pi \int_{x=a}^{x=b} f(x) \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx}$$

(ii) If Rotation about y-axis :—
curve $x = f(y)$



$$S = 2\pi \int x \, ds$$

$$\boxed{S = 2\pi \int_{y=a}^{y=b} f(y) \cdot \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy}$$

Parametric form of curve ⇒
let $x = \phi(t)$, $y = \psi(t)$ be the curve

2(i) About x-axis

$$S = 2\pi \int y \cdot ds$$

$$S = 2\pi \int_{t_1}^{t_2} y \cdot \frac{ds}{dt} \cdot dt$$

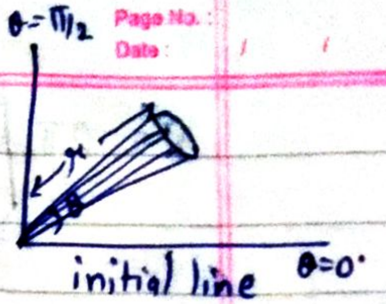
$$\left[\because \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \qquad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \right]$$

$$\boxed{S = 2\pi \int_{t_1}^{t_2} \psi(t) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt}$$

(ii). About y-axis

$$\boxed{S = 2\pi \int_{t_1}^{t_2} \phi(t) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt}$$

Polar Curve form ⇒
let $x = r\cos\theta$; $y = r\sin\theta$



(i) About $\theta = 0°$

$S = 2\pi \int y \cdot ds$

$$S = 2\pi \int_{\theta_1}^{\theta_2} r\sin\theta \cdot \frac{ds}{d\theta} \cdot d\theta \qquad \left[\because \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\right]$$

$$\boxed{S = 2\pi \int_{\theta_1}^{\theta_2} r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta}$$

Astroid

Cartesian :-

$\boxed{x^{2/3} + y^{2/3} = a^{2/3}}$

Parametric :-

$\boxed{x = a\cos^3 t}$

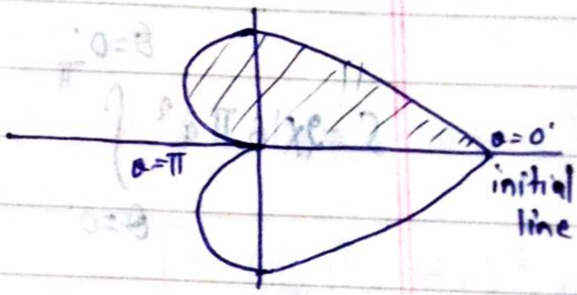
$\boxed{y = a\sin^3 t}$

(ii) About $\theta = \frac{\pi}{2}$

$$\boxed{S = 2\pi \int_{\theta_1}^{\theta_2} r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta}$$

Q.1 Find the surface of solid formed by the revalution of the cardioid $r = a(1+\cos\theta)$ about the initial line.

Ans - $r = a(1+\cos\theta)$



$\frac{dr}{d\theta} = -a\sin\theta$

$\theta = 0$ to $\theta = \pi$

$S = 2\pi \int y \cdot ds$

$$S = 2\pi \int_{\theta=0°}^{\pi} r\sin\theta \cdot \frac{ds}{d\theta} \cdot d\theta$$

$$S = 2\pi \int_{\theta=0°}^{\pi} r\sin\theta \cdot \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$S = 2\pi \int_{\theta=0^\circ}^{\pi} a(1+\cos\theta)\sin\theta \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}\cdot d\theta$$

$$S = 2\pi a^2 \int_{\theta=0^\circ}^{\pi} (1+\cos\theta)\sin\theta \sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\cdot d\theta$$

$$S = 2\pi a^2 \int_{\theta=0^\circ}^{\pi} (1+\cos\theta)\sin\theta \sqrt{2(1+\cos\theta)}\cdot d\theta$$

$$S = 2\pi a^2 \int_{\theta=0^\circ}^{\pi} (1+\cos\theta)\sin\theta \sqrt{2(2\cos^2\theta/2)}\cdot d\theta$$

$$S = 2\times 2\pi a^2 \int_{\theta=0^\circ}^{\pi} (1+\cos\theta)\sin\theta\,(2\cos\theta/2)\cdot d\theta$$

$$S = 2\times 4\pi a^2 \int_{\theta=0^\circ}^{\pi} 2\cos^2\theta/2\,(2\sin\theta/2\cos\theta/2)\,\cancel{\cos\theta}\,\cos\theta/2\cdot d\theta$$

$$S = 2\times 16\pi a^2 \int_{\theta=0^\circ}^{\pi} \sin\theta/2\,\cos^4\theta/2\cdot d\theta$$

$\Big\{$ let $\theta = 2t$, $d\theta = 2\,dt$, $\theta = 0.\ t=0$, $\theta=\pi,\ t=\pi/2$ $\Big\}$

$$S = 2\times 16\pi a^2 \int_{t=0^\circ}^{t=\pi/2} 2\sin t\,\cos^4 t\cdot dt$$

$$S = 2\times 32\pi a^2 \int_{0}^{\pi/2} \sin t\,\cos^4 t\cdot dt$$

$$S = 2\times 32\pi a^2\ \frac{\Gamma\frac{2}{2}\,\Gamma\frac{5}{2}}{2\,\Gamma\frac{7}{2}}$$

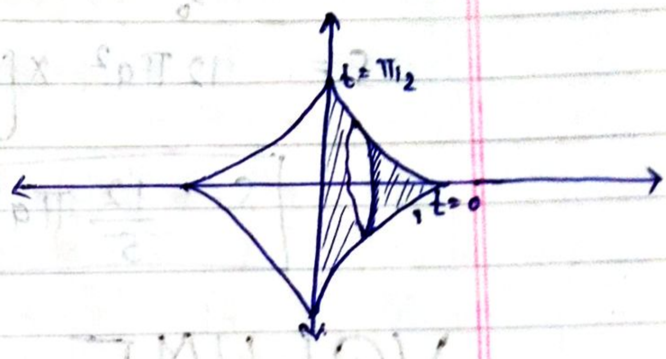
POPU

Page No.
Date:

$$S = 32\pi a^2 \frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \frac{64}{5}\pi a^2$$

Q2 Find the surface of the solid generated by the revolution of the <u>astroid</u> $x = a\cos^3 t$, $y = a\sin^3 t$ about x axis.

Ans - Astroid



$$x = a\cos^3 t$$

$$\frac{dx}{dt} = a(3\cos^2 t)(-\sin t)$$

$$\frac{dx}{dt} = -3a\sin t\cos^2 t$$

~~required surf~~ $y = a\sin^3 t$

$$\frac{dy}{dt} = 3a\sin^2 t\cos t$$

required surface

$$S = 2\pi\int y\,ds$$

$$S = 2\times 2\pi\int y\,ds \quad \text{(for astroid)}$$

$$S = 4\pi\int a\sin^3 t\,\frac{ds}{dt}\,dt \qquad \left[\because \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]$$

$$S = 4\pi\int_{t=0}^{\pi/2} a\sin^3 t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\cdot dt$$

$$S = 4\pi a\int_0^{\pi/2}\sin^3 t\sqrt{9a^2\sin^2 t\cos^4 t + 9a^2\sin^4 t\cos^2 t}\cdot dt$$

$$S = 4\pi a\int_0^{\frac{\pi}{2}}\sin^3 t\,(3a)(\sin t\cos t)\sqrt{\cos^2 t + \sin^2 t}\cdot dt$$

$$S = 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t \cdot dt$$

$\sin t = u$ $\qquad\qquad t = 0 \rightarrow u = 0$

$\cos t \cdot dt = du$ $\qquad\qquad t = \frac{\pi}{2} \rightarrow u = 1$

$$S = 12\pi a^2 \int_0^1 u^4 \cdot du$$

$$S = 12\pi a^2 \times \left[\frac{u^5}{5}\right]_0^1$$

$$\boxed{S = \frac{12}{5}\pi a^2}$$

VOLUME

Cartesian Curves

If curve $y = f(x)$

(i) Revolution about x axis

$$\boxed{V = \int \pi y^2 \, dx}$$

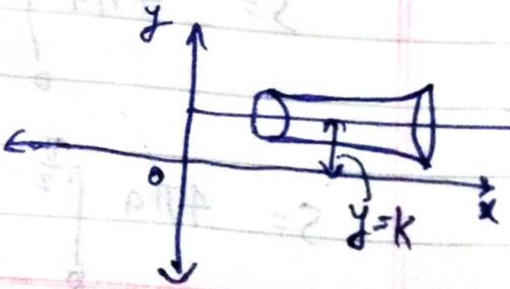
(ii) Revolution about y-axis (If $x = f(y)$)

$$\boxed{V = \int \pi x^2 \, dy}$$

$$\cancel{V = \int \pi x^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy}$$

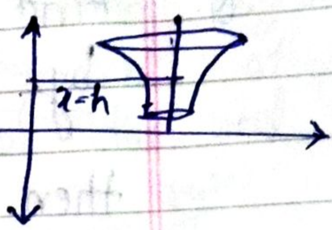
(iii) About any line $y = k$

$$\boxed{V = \int \pi (y-k)^2 \, dx}$$



(iv) About any line $x = h$

$$\boxed{V = \int \pi (x-h)^2 \, dy}$$



Parametric Curves

Curves $x = \phi(t)$, $y = \psi(t)$

(i) About x-axis

$$V = \int \pi y^2 \, dx$$

$$\boxed{V = \int \pi \, (\psi(t))^2 \left(\frac{dx}{dt}\right) \cdot dt}$$

(ii) About y-axis

$$V = \int \pi x^2 \, dy$$

$$\boxed{V = \int \pi \, (\phi(t))^2 \left(\frac{dy}{dt}\right) \cdot dt}$$

Polar Curve

If curve is of the form $r = f(\theta)$, then

(i) Revalution about $\theta = 0°$ (x axis)

$$V = \frac{2}{3}\pi \int_{r=r_1}^{r_2} r^3 \underline{\sin\theta} \cdot d\theta$$

(ii) Revolution about $\theta = \frac{\pi}{2}$ (y axis)

$$V = \frac{2}{3}\pi \int_{r=r_1}^{r_2} r^3 \underline{\cos\theta} \cdot d\theta$$

(iii) Revolution about any line $\theta = \alpha$

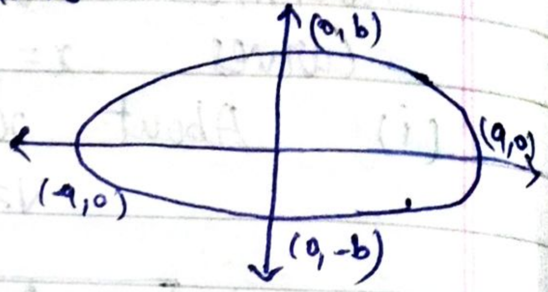
$$V = \frac{2}{3}\pi \int_{r_1}^{r_2} r^3 \underline{\sin(\theta - \alpha)} \cdot d\theta$$

Q. Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then. axes x & y.

about x - axis

Ans -

$$V = \int \pi y^2 \, dx$$



$$V = \int_{-a}^{a} \pi \, \frac{b^2}{a^2} (a^2 - x^2) \, dx$$

$$V = \frac{\pi b^2}{a^2} \int_{-a}^{a} (a^2 - x^2) \, dx$$

$$V = \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2) \cdot dx$$

$$V = \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$V = \frac{2\pi b^2}{\cancel{a^2}} \times \frac{2a^{\cancel{3}}}{3}$$

$$\boxed{V = \frac{4\pi b^2 a}{3}}$$

About y - axis

$$V = \int \pi x^2 \, dy$$

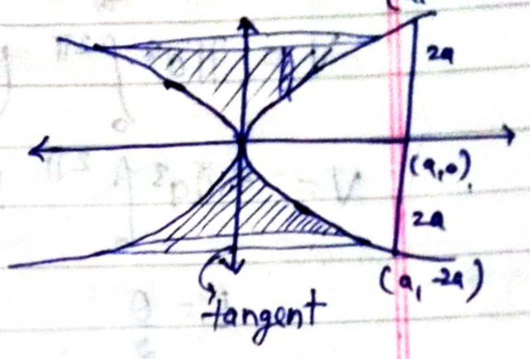
$$V = \int_{-b}^{b} \pi \, \frac{a^2}{b^2} (b^2 - y^2) \cdot dy$$

$$V = \frac{\pi a^2}{b^2} \times 2 \int_0^b (b^2 - y^2) \cdot dy$$

$$V = \frac{2\pi a^2}{b^2} \left[b^3 - \frac{b^3}{3} \right]$$

$$\boxed{V = \frac{4\pi a^2 b}{3}}$$

Q. The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.



Ans -

$$y^2 = 4ax$$

Volume about y-axis

$$V = \int \pi x^2 \, dy$$

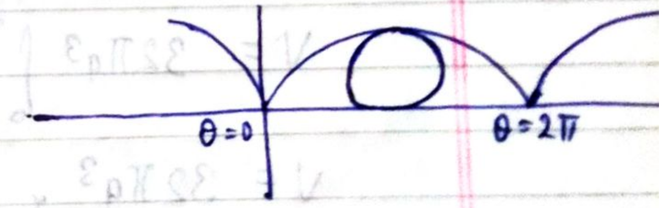
$$V = \int_{-2a}^{2a} \pi \left(\frac{y^2}{4a}\right)^2 dy = \frac{\pi}{16a^2} \times 2\int_0^{2a} y^4 \, dy$$

$$V = \frac{\pi}{8a^2} \times \frac{(2a)^5}{5} = \frac{4\pi a^3}{5} = \frac{4\pi a^3}{5}$$

Q. Find the volume of the solid generated by revolution of the $\boxed{\text{cycloid}}$

$$x = a(\theta - \sin\theta) \quad , \quad y = a(1-\cos\theta)$$

Ans - Equation of cycloid



$$x = a(\theta - \sin\theta)$$

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

$$y = a(1-\cos\theta) \Rightarrow \frac{dy}{d\theta} = \cancel{a\cos\theta} \; a\sin\theta$$

$$V = \int \pi y^2 \, dx$$

$$V = \int \pi \, a^2(1-\cos\theta)^2 \cdot \frac{dx}{d\theta} \cdot d\theta$$

$$V = \int_0^{2\pi} \pi a^2 (1-\cos\theta)^2 \, a(1-\cos\theta) \cdot d\theta$$

$$V = \pi a^3 \int_0^{2\pi} (1-\cos\theta)^3 \cdot d\theta$$

$$V = \pi a^3 \int_0^{2\pi} (2\sin^2\theta/2)^3 \, d\theta$$

$$V = 8\pi a^3 \int_0^{2\pi} \sin^6\frac{\theta}{2} \cdot d\theta$$

$$t = \frac{\theta}{2}, \quad dt = \frac{1}{2} d\theta, \quad t=0, \; t=\pi$$

$$V = 16\pi a^3 \int_{t=0}^{\pi} \sin^6 t \cdot dt$$

$$(\because \sin(\pi - t) = \sin t)$$

$$V = 32\pi a^3 \int_0^{\frac{\pi}{2}} \sin^6 t \cdot dt \quad (1)$$

$$\cancel{V = 32\pi a^3 \int_0^{\pi/2} \cos^6 t \cdot dt \quad (2)}$$

$$\cancel{2V = 32\pi a^3 \int_0^{\pi/2}}$$

$$V = 32\pi a^3 \int_0^{\pi/2} \sin^6 t \cos^0 t \cdot dt$$

$$V = 32\pi a^3 \times \frac{\Gamma\frac{7}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{8}{2}}$$

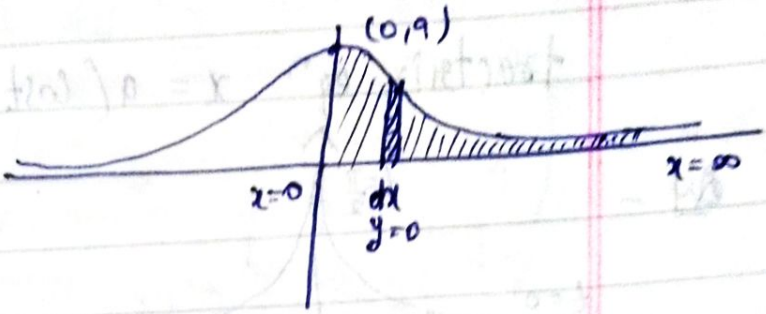
$$V = 32\pi a^3 \; \frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi} \times \sqrt{\pi}}{2 \times 3 \cdot 2 \cdot 1}$$

$$V = \frac{10\pi a^3 \sqrt{\pi}\sqrt{\pi}}{2}$$

$$V = 5\pi^2 a^3$$

Q. Prove that the volume of the solid generated by the revolution of the curve $y = \frac{a^3}{a^2+x^2}$ about its asymptote is $\frac{\pi^2 a^3}{2}$

Solution: We have



$$y = \frac{a^3}{a^2+x^2}$$

$$ya^2 + x^2y = a^3$$

$$x^2y = a^2(a-y)$$

$$x^2 = \frac{a^2(a-y)}{y}$$

If $y=0$, $x=\infty$

If $x=0$, $0 = \frac{a^2(a-y)}{y} \Rightarrow \boxed{a=y}$

(i) Curve cut y-axis at (0,a)

(ii) Curve does not pass through origin

(iii) Aysmptote $y=0$

Volume $= 2\int \pi y^2\,dx$

$$= 2\pi \int_0^\infty \left(\frac{a^3}{a^2+x^2}\right)^2 . dx$$

$$= 2\pi \int_0^\infty \frac{a^6}{(a^2+x^2)^2} . dx$$

$$= 2\pi a^6 \int_0^\infty \cancel{(a^2+x^2)^{-2}\,dx}\ \frac{1}{(a^2+x^2)^2} . dx$$

$x = a\tan\theta$, $dx = a\sec^2\theta$, $x=0$, $\theta=0$; $x=\infty$, $\theta=\frac{\pi}{2}$

$$= 2\pi a^6 \int_0^{\pi/2} \frac{a}{(a^2\sec^2\theta)^2} \sec^2\theta \cdot d\theta$$

$$2\pi a^3 \int_0^{\pi/2} \cos^2\theta\, d\theta = 2\pi a^3 \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} . d\theta$$

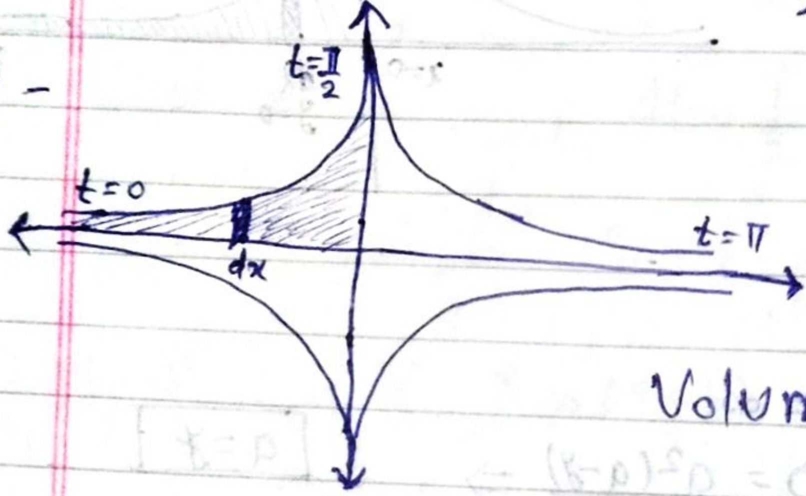
$$\pi a^3 \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \pi a^3\left(\frac{\pi}{2}\right)$$

$$= \pi^2 a^3/2$$

Q. Show that the volume of the solid generated by the revolution of the following tractrix about its asymptotes is $\frac{2}{3}\pi a^3$ and

tractrix eqn $x = a(\cos t + \frac{1}{2}\log \tan^2 t/2)$, $y = a\sin t$

Ans -



(i) Asymptote $y=0$

(ii) limit $t=0$ to $\frac{\pi}{2}$

Volume $= 2\int \pi y^2 dx$

$x = a(\cos t + \frac{1}{2}2\log \tan t/2)$

$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2} \times \sec^2 t/2 \times \frac{1}{2}\right)$

$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{2\frac{\sin t/2}{\cos t/2} \times \cos^2 t/2}\right)$

$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right)$

$\frac{dx}{dt} = a\left(\frac{1-\sin^2 t}{\sin t}\right) = a\frac{\cos^2 t}{\sin t}$

$V = 2\int_0^a \pi y^2 dx$

$V = 2\pi\int_0^{\pi/2} (a\sin t)^2 \frac{dx}{dt} \cdot dt$

$$V = 2\pi \int_0^{\pi/2} a^2 \sin^2 t \times a \frac{\cos^2 t}{\sin t} \, dt$$

$$V = 2\pi a^3 \int_0^{\pi/2} \sin t \cos^2 t \, dt \qquad t \to \pi/2,\ u = 0 \quad t \to 0,\ u = 1$$

$$\cos t = u \qquad \therefore -\sin t \, dt = du$$

$$V = -2\pi a^3 \int_1^0 u^2 \, du$$

$$V = 2\pi a^3 \times \left[\frac{u^3}{3}\right]_0^1 \Rightarrow \boxed{\frac{2\pi a^3}{3} = V} \qquad (H.P.)$$

$$S = \int 2\pi y \, ds$$

$$S = 2\pi \int (a \sin t) \left(\frac{ds}{dt}\right) dt$$

$$S = 2\pi \int_0^{\pi/2} (a \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

finally $\boxed{S = 4\pi a^2}$

Q. Show that the surface area of solid generated by revalution of the loop of the curve $x = t^2$, $y = t - \frac{t^3}{3}$ about x-axis is 3π & volume is $\frac{3\pi}{4}$

Ans- we have $x = t^2$; $y = t - t^3/3$

if $x = 0$, $t = 0$; if $y = 0$, ~~$t = 0$~~

$$\frac{3t - t^3}{3} = 0$$

$t = 0$ to $t = \sqrt{3}$

$$(3 - t^2)t = 0$$

$$t = 0,\ t = \sqrt{3}$$

$$\text{Surface} = \int 2\pi y \, ds$$

$$= \int 2\pi y \frac{ds}{dt} \, dt \qquad \left(\because \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right)$$

$$= \int_0^{\sqrt{3}} 2\pi\left(t-\frac{t^3}{3}\right)\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

$$\frac{dx}{dt} = 2t \quad , \quad \frac{dy}{dt} = 1 - \frac{\cancel{3}t^2}{\cancel{3}} = 1-t^2$$

$$= \int_0^{\sqrt{3}} 2\pi\left(t-\frac{t^3}{3}\right)\sqrt{4t^2+1-2t^2+t^4}\cdot dt$$

$$\int_0^{\sqrt{3}} 2\pi \; \frac{t(3-t^2)}{3}\sqrt{t^4+2t^2+1}$$

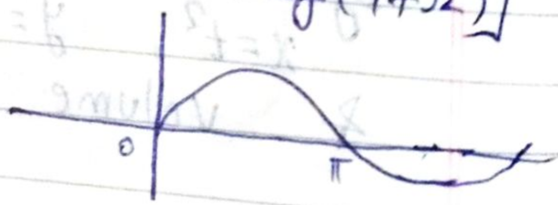
$$\int_0^{\sqrt{3}} 2\pi \; \frac{t(3-t^2)}{3}\times t\sqrt{t^2+2+\frac{1}{t^2}}$$

$$\left\{ \left(t^2+\frac{1}{t^2}\right) = \left(t+\frac{1}{t}\right)^2 - 2 \right\}$$

Q. Show that the surface of the solid obtained by revolving the arc of the curve $y = \sin x$; from $x = 0$ to $x = \pi$ about x-axis is $2\pi[\sqrt{2}+\log(1+\sqrt{2})]$

Ans – we have

$y = \sin x$



$$S = \int 2\pi y \, ds$$

$$S = 2\pi\int y \frac{ds}{dx}\,dx \quad \left[\because \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\right]$$

$$S = 2\pi\int_0^{\pi} \sin x\sqrt{1+\cos^2 x}\cdot dx$$

$\cos x = t \Rightarrow -\sin x\, dx = dt$

$x = 0 \rightarrow t = 1 \quad , \quad x = \pi \rightarrow t = -1$

$$S = -2\pi \int_{1}^{-1} \sqrt{1+t^2}\cdot dt$$

$$S = 2\pi \int_{-1}^{1} \sqrt{1+t^2}\cdot dt = 4\pi \int_{0}^{1} \sqrt{1+t^2}\cdot dt$$

$$S = 4\pi \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left| t+\sqrt{1+t^2} \right| \right]_0^1$$

$$S = 4\pi \left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\log(1+\sqrt{2}) - 0 - \frac{1}{2}\log 1 \right]$$

$$S = 2\pi \left[\sqrt{2} + \log(1+\sqrt{2}) \right] \quad \text{(H.P.)}$$

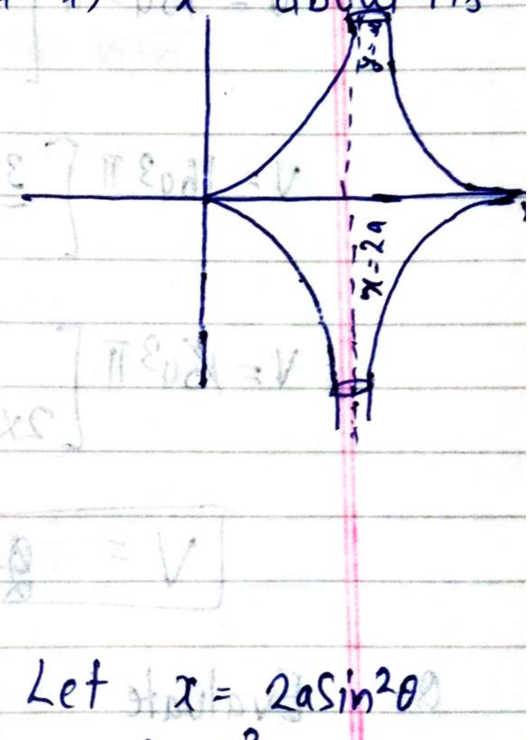
Q. Find the volume of solid generated by revolution of the cissoid $y^2(2a-x) = x^3$ about its asymptote.

Ans- we have

$$y^2(2a-x) = x^3$$

$$y^2 = \frac{x^3}{2a-x}$$

At $x = 2a$, $y = \infty$



$$V = \int \pi x^2 \, dy$$

$$V = \int_{-\infty}^{\infty} \pi (2a-x)^2 \, dy$$

~~$V = \int_{-\infty}^{\infty} \pi$~~

Let $x = 2a\sin^2\theta$

$$y^2 = \frac{x^3}{2a-x}$$

$$y^2 = \frac{8a^3\sin^6\theta}{2a\cos^2\theta} = \frac{4a^2\sin^6\theta}{\cos^2\theta}$$

$$y = \frac{2a\sin^3\theta}{\cos\theta} = \cancel{2a\tan\theta\sin^2\theta}$$

~~$\frac{dy}{d\theta} = 2a\left[\tan\theta\sin 2\theta + \frac{\sin^2\theta}{\cos^2\theta}\right]$~~

$$\frac{dy}{d\theta} = 2a\left[\frac{3\sin^2\theta\cos^2\theta + \sin^4\theta}{\cos^2\theta}\right]$$

$$V = 2\int_{\theta=0}^{\frac{\pi}{2}} \pi\,(2a - 2a\sin^2\theta)^2 \cdot \frac{dy}{d\theta}\cdot d\theta$$

$$V = 2\pi\int_{\theta=0}^{\frac{\pi}{2}} 4a^2\cos^4\theta \times 2a\left(\frac{3\sin^2\theta\cos^2\theta + \sin^4\theta}{\cos^2\theta}\right)\cdot d\theta$$

$$V = 16a^3\pi\int_{\theta=0}^{\frac{\pi}{2}} 3\sin^2\theta\cos^4\theta + \sin^4\theta\cos^2\theta \cdot d\theta$$

$$V = 16a^3\pi\left[3\left(\frac{\Gamma\frac{3}{2}\,\Gamma\frac{5}{2}}{2\Gamma 4}\right) + \frac{\Gamma\frac{5}{2}\,\Gamma\frac{3}{2}}{2\Gamma 4}\cdot d\theta\right]$$

$$V = 16a^3\pi\left[\frac{3\times\frac{1}{2}\times\sqrt{\pi}\times\frac{\cancel{3}}{2}\times\frac{1}{2}\sqrt{\pi}}{2\times\cancel{3}\times2\times1} + \frac{\frac{\cancel{3}}{2}\times\frac{1}{2}\times\frac{1}{2}\times\pi}{2\times\cancel{3}\times2\times1}\right]$$

$$V = \cancel{16}a^3\pi\left[\frac{3}{2\times\cancel{16}}\pi + \frac{\pi}{2\times\cancel{16}}\right]$$

$$\boxed{V = 2a^3\pi^2}$$

Q. Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^4}}\cdot dx$

Ay- let $x^2 = \sin\theta$

$2x\,dx = \cos\theta\cdot d\theta$ $\qquad x=0,\ \theta=0$

$dx = \frac{\cos\theta\cdot d\theta}{2\sqrt{\sin\theta}}$ $\qquad x=1,\ \theta=\pi/2$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2\theta}}\cdot\frac{\cos\theta}{2\sqrt{\sin\theta}}\cdot d\theta$$

POPU

$$\frac{1}{2}\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta \cdot d\theta$$

$$= \frac{1}{2} \frac{\Gamma\frac{-\frac{1}{2}+1}{2}\,\Gamma\frac{0+1}{2}}{2\Gamma\frac{-\frac{1}{2}+0+2}{2}} = \frac{1}{4}\frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{4}}$$

$$\frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{4\,\Gamma\left(1-\frac{1}{4}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{4\,\Gamma\left(\frac{3}{4}\right)} \quad \text{Ans -}$$

Q. Evaluate $\Gamma\left(-\frac{1}{2}\right)$

Ans - we know $\Gamma(n+1) = n\Gamma n$

$$\Gamma\left(-\frac{1}{2}+1\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$$

$$\Gamma\frac{1}{2} = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$$

$$-2\sqrt{\pi} = \Gamma\left(-\frac{1}{2}\right)$$

Q. Evaluate $\Gamma\left(-\frac{3}{2}\right)$

Ans - we know $\Gamma(n+1) = n\Gamma n$

$$\Gamma\left(-\frac{3}{2}+1\right) = -\frac{3}{2}\Gamma\left(-\frac{3}{2}\right)$$

$$\frac{-2\sqrt{\pi}\times 2}{-3} = \Gamma\left(-\frac{3}{2}\right)$$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3}$$

Q. $\int_0^\infty \frac{1}{1+x^4} \cdot dx$

Ans - $x^2 = \tan\theta \Rightarrow 2x \cdot dx = \sec^2\theta \cdot d\theta$

$x = 0, \quad \theta = 0$

$x = \infty, \quad \theta = \pi/2$

$$\int_0^{\infty} \frac{1}{1+(x^2)^2} \cdot dx$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cancel{1+\tan^2\theta}} \cdot \frac{\cancel{\sec^2\theta}}{2\sqrt{\tan\theta}} \cdot d\theta$$

$$\frac{1}{2}\int_0^{\pi/2} \cos^{1/2}\theta \, \sin^{-1/2}\theta \cdot d\theta$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{4}$$

$$\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}{4} = \frac{\pi}{4\sin\frac{\pi}{4}} = \frac{\pi}{4 \times \frac{1}{\sqrt{2}}}$$

$$= \frac{\pi}{2\sqrt{2}}$$

Chapter - 3
Fourier Series

$\boxed{\sin n\pi = 0}$, where $n \in Z \rightarrow$ integer $(\ldots -2, -1, 0, 1, 2 \ldots)$

(i) $n=0$, $\sin 0 = 0$ (ii) $n=1$, $\sin \pi = 0$, (iii) $n=2$, $\sin 2\pi = 0$

$\boxed{\cos n\pi = (-1)^n}$

(i) $n=1$, $\cos \pi = (-1)^1 = -1$
(iii) $n=3$, $\cos 3\pi = (-1)^3 = -1$ (ii) $n=2$, $\cos 2\pi = (-1)^n = 1$

$$\int uv \, dx = u \int v \, dx - \int \left\{ \frac{du}{dx} \int v \, dx \right\} dx$$

$$\Rightarrow \int uv \, dx = U(Iv) - D(U)(I^2 v) + D^2(U)(I^3 v) - D^3(U)(I^4 v) + \cdots \quad (0) \qquad \left[\begin{array}{l} \int \Rightarrow I \\ \frac{d}{dx} \Rightarrow D \end{array} \right]$$

Periodic function :- A function $f(x)$ which satisfies the relation $f(x+T) = f(x)$ for all real x is called periodic function.

$$f(x+T) = f(x+2T) = \cdots = f(x+nT) = f(x)$$

eg- $\sin x$, $\cos x$

Fourier Series $\Rightarrow$

Definition $\Rightarrow$ let $f(x)$ be a periodic function with period 2π defined in the interva $\alpha < x < \alpha + 2\pi$ then the infinite series is given by

$\to$ $\sin n\pi \Rightarrow \pi, 2\pi, 0$ $\to$ (0)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——(1)}$$

or

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0, a_n and b_n known fourier coefficients

$$a_0 = \frac{1}{2\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cdot dx \quad \text{——(2)}$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx \quad \text{——(3)}$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx \quad \text{——(4)}$$

Case-I If $\alpha = 0$ $(0, 2\pi)$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \cdot dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \cdot dx$$

Case-II If $\alpha = -\pi$ $(-\pi, \pi)$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \cdot dx$$

Q.1 Find the fourier series for the function

$$f(x) = \begin{cases} \pi + x, & -\pi \text{ to } 0 \quad -\pi < x < 0 \\ \pi - x, & 0 \text{ to } \pi \quad 0 \leq x < \pi \end{cases}$$

Ans Given internal $(-\pi, \pi)$

By definition of fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---}(1)$$

when

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} (\pi + x)\,dx + \int_{0}^{\pi} (\pi - x)\cdot dx\right]$$

$$a_0 = \frac{1}{2\pi}\left[\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}\right]$$

$$a_0 = \frac{1}{2\pi}\left(+\pi^2 - \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right)$$

$$a_0 = \frac{\pi}{2} \quad \text{---}(2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \cdot dx$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x) \cos nx \cdot dx + \int_{0}^{\pi} (\pi - x) \cos nx \cdot dx\right]$$

$$\cancel{a_n = \frac{1}{\pi}\left[\left[\frac{\pi \sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \left[\frac{\pi \sin nx}{n}\right]_{0}^{\pi}\right.}$$

$$\left. - \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi}\right]$$

$$a_n = \frac{1}{\pi}\left[\left[(\pi + x)\frac{\sin nx}{n} - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \left((\pi - x)\frac{\sin nx}{n}\right)_{0}^{\pi}\right.$$

$$- \left((-1)\left(\frac{-\cos nx}{n^2}\right)\right)_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\left(\pi(0) + \frac{1}{n^2} - (0) - \frac{\cos(-n\pi)}{n^2}\right) + 0 - 0 - \left(\frac{(-1)^n}{n^2} + \frac{-1}{n^2}\right)\right.$$

$$\frac{1}{\pi}\left[0 + \frac{1}{n^2} + \frac{1}{n^2} - 2\frac{(-1)^n}{n^2}\right)$$

$$a_n = \frac{2}{\pi n^2}\left(1 - (-1)^n\right) \quad \text{--- (3)}$$

or

$$a_n = \begin{cases} 0 & , \text{ when } n = \text{even} \\ \frac{4}{\pi n^2} & , \text{ when } n = \text{odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \cdot dx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi + x)\sin nx \cdot dx + \int_{0}^{\pi}(\pi - x)\sin nx \cdot dx\right.$$

$$b_n = \frac{1}{\pi}\left[\left\{-(\pi+x)\frac{\cos nx}{n} - (1)\frac{\sin nx}{n^2}\right\}_{-\pi}^{0} + \left\{(-1)(\pi - x)\frac{\cos nx}{n} - (-1)\frac{\sin nx}{n^2}\right\}_{0}^{\pi}\right.$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi}{n} + \frac{(\pi - 0)}{n}\right] = 0 \quad \text{--- (4)}$$

Now putting the value of a_0, a_n & b_n in eq^n (1)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left(1-(-1)^n\right)\cos nx + 0$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left[\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2}\right)\cos nx\right]$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{2}{1^2}\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x + \cdots\right]$$

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Even & odd function

If $f(-x) = f(x) \longrightarrow$ Even function

If $f(-x) = -f(x) \longrightarrow$ odd function

For even function, Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad ; \quad \boxed{b_n = 0}$$

x^3, $(-x)^3 = -x^3$

For odd function, Fourier series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Q. Obtain the Fourier Series for the function $f(x) = x^2$, $-\pi < x < \pi$ and denote the following relation from

(i) $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

(ii) $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

(iii) $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

POPU

Page No.:
Date:

Ay - Since $f(x) = x^2$ is an even function, then Fourier Series

$$x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—(1)} \quad , (\because b_n = 0)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \cdot dx = \frac{1}{2\pi}\times 2\int_{0}^{\pi} x^2\cdot dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3} \quad \text{—(2)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx \cdot dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \cdot dx$$

$$a_n = \frac{1}{\pi}\times 2\int_{0}^{\pi} x^2 \cos nx \cdot dx$$

$$a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[2x\frac{\cos nx}{n^2}\right]_0^{\pi} + \left[\frac{x^2\sin nx}{n} - \frac{2\sin nx}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{n^2\cancel{\pi}}\left[2\cancel{\pi}\cos n\pi\right] = \frac{4}{n^2}\cos n\pi$$

$$a_n = \frac{4}{n^2}(-1)^n \quad \text{—(3)}$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$

$$x^2 = \frac{\pi^2}{3} + \left(-\frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \frac{4}{4^2}\cos 4x - \cdots\right)$$

$$x^2 = \frac{\pi^2}{3} + 4\left[\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \cdots\right]$$

Now put $x=0$

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\frac{\pi^2}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \quad \text{(H.P.)} \quad \text{—(5)}$$

Now put $x=\pi$ $\{\cos\pi = -1\}$ $\{\cos 2\pi = 1\}$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{-(-1)}{1^2} + \frac{(1)}{2^2} - \frac{(-1)}{3^2} + \frac{(1)}{4^2} + \cdots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{—(6)} \quad \text{(H.P.)}$$

Additing (5) & (6)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right) + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\frac{\pi^2}{4} = 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \quad \text{(H.P)}$$

Q. Expand $f(x) = |\cos x|$ in a fourier series in the integral $(-\pi, \pi)$

Ans- Since $f(x) = |\cos x|$ is an even function, then fourier series

Page No. :
Date :

$$|\cos x| = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—(1)}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\cos x| \cdot dx$$

$$a_0 = \frac{1}{2\pi} \times 2\int_{0}^{\pi} |\cos x| \cdot dx$$

$$a_0 = \frac{1}{\pi}\int_{0}^{\pi} |\cos x| \cdot dx \qquad \left\{ |\cos x| = \begin{cases} \cos x & 0 < x < \pi/2 \\ -\cos x & \frac{\pi}{2} < x < \pi \end{cases} \right.$$

$$a_0 = \frac{1}{\pi}\left[\int_{0}^{\pi/2} \cos x \cdot dx + \int_{\pi/2}^{\pi} -\cos x \cdot dx\right]$$

$$a_0 = \frac{1}{\pi}\left[(\sin x)_0^{\pi/2} + (-\sin x)_{\pi/2}^{\pi}\right]$$

$$a_0 = \frac{1}{\pi}\left[1 - 0 - (0 - 1)\right] = \frac{2}{\pi}$$

$$a_0 = \frac{2}{\pi} \quad \text{—(2)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \cdot dx$$

$$a_n = \frac{1}{\pi} \times 2\int_{0}^{\pi} |\cos x| \cos nx \cdot dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} |\cos x| \cos nx \cdot dx$$

$$a_n = \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos x \cos nx \cdot dx - \int_{\pi/2}^{\pi} \cos x \cos nx \cdot dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{0}^{\pi/2} 2\cos x \cos nx \cdot dx - \int_{\pi/2}^{\pi} 2\cos x \cos nx \cdot dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{0}^{\pi/2}(\cos(nx+x) + \cos(nx-x))dx - \int_{\pi/2}^{\pi} \cos(nx+x) + \cos(nx-x)\right.$$

Page No.:
Date:

$$a_n = \frac{1}{\pi}\left[\left(\frac{\sin(nx+x)}{n+1} + \frac{\sin nx - x}{n-1}\right)_0^{\pi/2} - \left(\frac{\sin nx+x}{n+1} + \frac{\sin nx-x}{n-1}\right)_{\pi/2}^{\pi}\right]$$

$$a_n = \frac{1}{\pi}\left[\left[\frac{\sin x(n+1)}{n+1} + \frac{\sin x(n-1)}{n-1}\right]_0^{\pi/2} - \left[\frac{\sin(x)(n+1)}{n+1} + \frac{\sin x(n-1)}{n-1}\right]_{\pi/2}^{\pi}\right]$$

$$a_n = \frac{1}{\pi}\left[\left(\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right) + \left(\frac{\sin\frac{\pi}{2}(n+1)}{n+1} + \frac{\sin\frac{\pi}{2}(n-1)}{n-1}\right)\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{2\sin(n+1)\pi/2}{n+1} + \frac{2\sin(n-1)\pi/2}{n-1}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)}{n+1} + \frac{\sin(n\pi/2 - \pi/2)}{n-1}\right]$$

$$\{\sin(90-\theta) = \cos\theta,\ \sin(90+\theta) = \cos\theta\}$$

$$a_n = \frac{2}{\pi}\left[\frac{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)}{n+1} - \frac{\sin(\pi/2 - n\pi/2)}{n-1}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right]$$

$$a_n = \frac{2}{\pi}\cos\frac{n\pi}{2}\left[\frac{n-1-n-1}{n^2-1}\right]$$

$$a_n = \frac{2}{\pi}\cos\frac{n\pi}{2}\left(\frac{-2}{n^2-1}\right)$$

$$a_n = \frac{-4}{\pi}\cos\frac{n\pi}{2}\left(\frac{1}{n^2-1}\right),\ n \neq 1 \quad (3)$$

Now putting value of a_0 & a_n in eqn -(1)

$$|\cos x| = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left(\frac{-4}{\pi}\right)\cos\frac{n\pi}{2}\left(\frac{1}{n^2-1}\right)\cos nx$$

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \cos\frac{n\pi}{2}\left(\frac{1}{n^2-1}\right)\cos nx$$

$$\cancel{|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\Big[}$$

For $n=1$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| \cos x \cdot dx$$

$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x \cdot dx - \int_{\pi/2}^{\pi} \cos^2 x \cdot dx\right]$$

$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right) \cdot dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right) \cdot dx\right]$$

$$a_1 = \frac{1}{\pi}\left[\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} - \left[x + \frac{\sin 2x}{2}\right]_{\pi/2}^{\pi}\right]$$

$$a_1 = \frac{1}{\pi}\left[\frac{\pi}{2} + 0 - \left(\pi + 0 - \frac{\pi}{2}\right)\right]$$

$$a_1 = \frac{1}{\pi}(0) = 0$$

So

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \cos\frac{n\pi}{2}\left(\frac{1}{n^2-1}\right)\cos nx$$

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\left[0 - \frac{\cos 2x}{3} + \frac{1}{15}\cos 4x \cdots\right]$$

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{\cos 2x}{3} - \frac{1}{15}\cos 4x + \cdots\right)$$

Half Range Series $\Rightarrow$

eg- $-\pi < x < \pi$ → full range

$0 < x < \pi$ → Half range

POPU

Page No.:
Date:

Half Range Fourier Series ⇒

Suppose range $0 < x < l$

$$f(x) = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)}_{\text{Cosine Series}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)}_{\text{Sine series}}$$

Cosine Series :—

$$a_0 = \frac{1}{l} \int_0^l f(x) \cdot dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) \cdot dx \quad ; \quad b_n = 0$$

Half range Sine series

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) \cdot dx$$

Q. Find half range Cosine Series for the function $f(x) = (x-1)^2$, $0 < x < 1$

hence show that

$$\pi^2 = 8\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

Ans- Given $f(x) = (x-1)^2$, $0 < x < 1$

$l = 1$

$$a_0 = \frac{1}{l} \int_0^l f(x) \cdot dx$$

$$a_0 = \frac{1}{1} \int_0^1 (x-1)^2 \cdot dx$$

$$a_0 = \left[\frac{(x-1)^3}{3}\right]_0^1 = 0 - \frac{(-1)}{3} = \frac{1}{3} \quad \text{—(1)}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) \cdot dx$$

$$a_n = 2\int_0^1 (x-1)^2 \cos n\pi x \cdot dx$$

$$a_n = 2\left[(x-1)^2 \frac{\sin n\pi x}{n\pi} - 2(x-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right) - 0 \right]_0^1$$

$$a_n = 2\left[0 + 0 + 0 - 0 + 2(+1)\frac{\cos n\pi}{n^2\pi^2} \right]$$

$$a_n = \frac{4\cos n\pi(0)}{n^2\pi^2} \quad \text{---} \; (2)$$

For a_0 $\quad a_n = \dfrac{4}{n^2\pi^2}$ — (2)

For Cosine Series

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2}$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2} \left[\sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2} \right.$$

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2} \quad \text{---} \; (3)$$

putting $x = 0$

$$(1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots \right] \cos 0$$

$$\frac{2}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right] \quad \text{---} \; (4)$$

putting $x = 1$ in eqⁿ – (3)

$$0 = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right] \quad (5)$$

Now eqⁿ (4) – (5)

$$\frac{2}{3} - \left(\frac{-1}{3}\right) = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\pi^2 = 8\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] \quad \text{(H.P.)}$$

Q. Find the half range sine series for $f(x) = x(\pi - x)$, in $0 < x < \pi$ and hence find the sum of series

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

1- Given $f(x) = x(\pi - x)$ $\quad 0 < x < \pi$

$l = \pi$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right).dx$$

$$b_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin\left(\frac{n\pi x}{\pi}\right).dx$$

$$b_n = \frac{2}{\pi}\int_0^\pi x\pi\sin(nx).dx - \frac{2}{\pi}\int_0^\pi x^2\sin(nx).dx$$

$$b_n = \frac{2}{\pi}\left[\pi\int_0^\pi x\sin nx.dx - \int_0^\pi x^2\sin nx.dx\right.$$

$$b_n = \frac{2}{\pi}\left[\pi\left(\left(-x\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right) - \left[\left(-x^2\frac{\cos nx}{n}\right)\right.\right.$$

$$\left.\left. - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(-\frac{\cos nx}{n^3}\right)\right]\right]_0^\pi$$

$$b_n = \frac{2}{\pi}\left[-\pi x\frac{\cos nx}{n} + 2\frac{\cos nx}{n^3}\right]_0^\pi$$

$$b_n = \frac{2}{\pi}\left[-\pi^2\frac{(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$

$$b_n = \frac{-2}{\pi}\left[\frac{2((-1)^n-1)}{n^3} \cancel{- \frac{\pi^2(-1)^n}{n^2}}\right]$$

$$b_n = \frac{-2}{\pi}\cdot\frac{2}{n^3}[(-1)^n-1]$$

$$b_n = \frac{4}{\pi n^3}[1-(-1)^n] \quad \text{—— (1)}$$

$$\because \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$l \to \pi$

$$x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}[1-(-1)^n] \sin\frac{n\pi x}{\pi}$$

$$\pi x - x^2 = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}[1-(-1)^n] \sin nx$$

$x = \frac{\pi}{2}$ put now

$$\pi \times \frac{\pi}{2} - \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}[1-(-1)^n] \sin\frac{n\pi}{2}$$

$$\frac{\pi^2}{4} = \frac{4}{\pi}\left[\frac{2}{1^3} + 0 + \frac{2}{3^3} + 0 + \frac{2}{5^3} + \cdots\right]$$

$$\frac{\pi^3}{16} = 2\left[\frac{\sin\pi/2}{1^3} \frac{\sin 3\pi/2}{3^3} + \frac{\sin 5\pi/2}{5^3} + \cdots\right]$$

$$\frac{\pi^3}{3.2} = \left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\right)$$

Q. Find half range consine series for the following function $f(x) = x\sin x$, $0 < x < \pi$

Ans – Half range consine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{—— (1)}$$

Date :

where $a_0 = \frac{1}{l}\int_0^l f(x)\cdot dx$ $(\because l = \pi)$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\sin x\cdot dx$$

$$a_0 = \frac{1}{\pi}\left[x(-\cos x) - (-\sin x)\right]_0^{\pi}$$

$$a_0 = \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi} = \frac{-1}{\pi}\left[x\cos x\right]_0^{\pi}$$

$$a_0 = \frac{-1}{\pi}[-\pi] = 1 \quad \text{—— (2)}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)\cdot dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos\left(\frac{n\pi x}{\pi}\right)\cdot dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\cdot dx$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\,(2\cos nx\sin x)\cdot dx \quad \left[\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)\right]$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\sin(nx+x) - x\sin(nx-x)\cdot dx$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x - \int_0^{\pi} x\sin(n-1)x\cdot dx$$

$$a_n = \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - (1)\frac{(-\sin(n+1)x)}{(n+1)^2}\right]_0^{\pi}$$
$$- \left[x\frac{(-\cos(n-1)x}{(n-1)} - (1)\frac{(-\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[-x\frac{\cos(n+1)x}{n+1} + \frac{x\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{n+1} + \frac{\pi\cos(n-1)\pi}{n-1}\right]$$

$$a_n = -\frac{\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{(n-1)}$$

$$a_n = \frac{-\cos(\pi+n\pi)}{(n+1)} + \frac{\cos(\pi-n\pi)}{n-1}$$

$$a_n = \frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1}$$

$$a_n = \frac{n\cos n\pi - \cos n\pi - n\cos n\pi - \cos n\pi}{n^2-1}$$

$$a_n = \frac{-2(\cos n\pi)}{n^2-1}$$

$$a_n = \frac{-2(-1)^n}{n^2-1} \quad n \neq 1 \qquad (3)$$

Now putting value of a & a_n in eq^n (1)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n^2-1} \cos\left(\frac{n\pi x}{l}\right)$$

$$x\sin x = 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-1} \cos nx \qquad (4)$$

~~$x\sin x = 1 - 2\phi$~~

for $a = 1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \cdot dx$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \cos x \sin x \cdot dx$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \cdot dx$$

$$a_1 = \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$a_1 = \frac{1}{\pi}\left[\frac{-x\cos 2x}{2}\right]_0^{\pi}$$

$$a_1 = \frac{-1}{2\pi}\left[\pi\cos 2\pi - 0\right] = \frac{-1}{2} \quad (5)$$

In eqⁿ (4)

$$x\sin x = 1 - \frac{1}{2} - 2\left[\frac{1}{3}\cos 2x - \frac{1}{8}\cos 3x + \cdots\right]$$

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{1}{3}\cos 2x - \frac{1}{8}\cos 3x + \cdots\right]$$

Dirichlet's condition ⇒

Dirichlet's conditions are (i) $f(x)$ is finite & single value in $(\alpha, \alpha+2\pi)$

(ii) $f(x)$ is periodic

(iii) $f(x)$ & $f'(x)$ are piecewise conditions in $(\alpha, \alpha+2\pi)$, so that the integration in the given interval is possible.

Thus the Fourier series with a_0, a_n & b_n converge to

(a) $f(x)$ if x is a pt of continuity

(b) $\frac{1}{2}[f^+(x) + f^-(x)]$, if x is a point of discontinuity where $f^+(x)$ is the RHL & $f^-(x)$ is the LHL of $f(x)$ of x.

Parsevel's Theorem and Identities ⇒ If the fourier series of the function $f(x)$. Over, an interval $\alpha < x < \alpha + 2\pi$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

then

$$\frac{1}{l}\int_{\alpha}^{\alpha+2\pi}[f(x)]^2 \cdot dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

proof:- By defination of Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{—— (1)}$$

where $a_0 = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x) \cdot dx$ —— (2)

$$a_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) \cdot dx \quad \text{—— (3)}$$

$$b_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) \cdot dx \quad \text{—— (4)}$$

Now multiplying (1) by $f(x)$, we get

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n f(x) \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \int_{\alpha}^{\alpha+2l}(f(x))^2 \cdot dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2l} f(x) \cdot dx + \int_{\alpha}^{\alpha+2l}\sum_{n=1}^{\infty} a_n f(x) \cos\left(\frac{n\pi x}{l}\right) dx + \int_{\alpha}^{\alpha+2l}\sum_{n=1}^{\infty} b_n f(x) \sin\left(\frac{n\pi x}{l}\right) \cdot dx$$

$$\Rightarrow \int_{\alpha}^{\alpha+2l}[f(x)]^2 \cdot dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2l} f(x) \cdot dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) \cdot dx$$

Page No.
Date :

from eq^n 1, 2, 3 & 4

$$\int_{\alpha}^{\alpha+2l} [f(x)]^2 . dx = \frac{a_0}{2}(l a_0) + \sum_{n=1}^{\infty} a_n(l a_n) + \sum_{n=1}^{\infty} b_n(l b_n)$$

$$= \int_{\alpha}^{\alpha+2l} [f(x)]^2 . dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

$$\frac{1}{l}\int_{\alpha}^{\alpha+2l} [f(x)]^2 . dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

(H.P.)

Q. Find Fourier Series expansion of x^2 in $(-\pi, \pi)$ use parseval's identity to prove that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

Sol- We have $f(x) \to x^2$ $\to$ even function

$\because b_n = 0$

By definition fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) . dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx . dx = \frac{4}{n^2}(-1)^n$$

By parseval's Identity

$$\frac{1}{l}\int_{\alpha}^{\alpha+2\pi} [f(x)]^2 . dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$= \frac{4}{2}\left(\frac{\pi^2}{3}\right)^2 \times \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\right)^2$$

$$= \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}(-1)^{2n}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(x^2)^2 dx = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}(-1)^{2n}$$

$$\frac{1}{\pi}\times 2\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{2\pi^4}{9} + 16\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]$$

$$\frac{2}{\pi}\times\frac{\pi^5}{5} = \frac{2\pi^4}{9} + 16\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]$$

$$\frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 16\left[1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]$$

$$\frac{18\pi^4 - 10\pi^4}{45}\times\frac{1}{16} = 1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$$

$$\frac{8\pi^4}{45}\times\frac{1}{16} = 1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$$

$$\frac{\pi^4}{90} = 1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$$

Q.1 Find the Fourier series to represent $f(x) = |x|$ for $-\pi < x < \pi$

Q.2 Find the fourier series to represent $f(x) = x\cos x$ $-\pi < x < \pi$

Q.3 Find the fourier series expansion for the function given as

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } 0 < x \le \pi \end{cases}$$

Hence prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

Page No.
Date: / /

4- By defination of fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{—①}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -1\cdot dx + \int_{0}^{\pi} 1\cdot dx\right]$$

$$a_0 = \frac{1}{2\pi}\left([-x]_{-\pi}^{0} + [x]_{0}^{\pi}\right)$$

$$a_0 = \frac{1}{2\pi}(-\pi + \pi) = 0 \quad \text{—②}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\cdot dx$$

$$a_n = \frac{1}{\pi}\left[\left[\int_{-\pi}^{0} -1\cdot\cos nx\cdot dx\right] + \left[\int_{0}^{\pi} 1\cdot\cos nx\, dx\right]\right]$$

$$a_n = \frac{1}{\pi}\left[-\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right]$$

$$a_n = \frac{1}{\pi}\left[-\left(+\frac{\sin n\pi}{n}\right) + \frac{\sin n\pi}{n}\right] = 0 \quad \text{—③}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\cdot dx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -1\cdot\sin nx\cdot dx + \int_{0}^{\pi} 1\cdot\sin nx\cdot dx\right]$$

$$\frac{1}{\pi}\left[+\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} - \left[\frac{\cos nx}{n}\right]_{0}^{\pi}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{1}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + \frac{1}{n}\right]$$

$$b_n = \frac{2}{n\pi} - \frac{2(-1)^n}{n\pi} = \frac{2}{n}\left(\frac{1}{\pi} - \frac{(-1)^n}{\pi}\right) \quad \text{—④}$$

in eq^n - (1)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$x = \pi/2$

$$1 = 0 + 0 + \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-(-1)^n \sin\frac{n\pi}{2}$$

$$1 = \frac{2}{\pi}\left[\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\frac{n\pi}{2}\right]$$

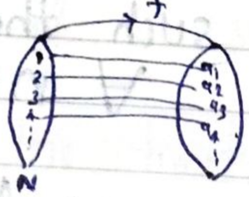
$$1 = \frac{2}{\pi}\left[\frac{2}{1} - \frac{2}{3} + 0 + \frac{1}{5} - \frac{1}{7}\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - - - - -$$

Chapter - 2

Sequence & Series

Sequence :- A sequence is a function whose domain is the set of natural no. $N \in A$ and any range may be any set A.



Sequence $a_1, a_2, a_3, \ldots a_n$ is denoted by $\{a_n\}$

Real sequence :- Real Sequence is defined as $f: N \to R$ where range of a function is real and domain as natural number.

Monotonic Sequence :- A sequence $\{x_n\}$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

Monotonic sequence

- Monotonically increasing
- Monotonically decreasing

(i) **Monotonically increasing sequence :-** A sequence $\{x_n\}$ is said to be monotically increasing if $x_n \leq x_{n+1}$, $\forall\ n \in N$

or

$x_1 \leq x_2 \leq x_3 \ldots x_n \leq x_{n+K} \Rightarrow x_n \leq x_{n+1}\ \forall\ n \in N$

(ii) **Monotonically decreasing sequence :-** A sequence $\{x_n\}$ is said to be monotically decreasing if $x_n \geq x_{n+1}$, $\forall\ n \in N$

or $x_1 \geq x_2 \geq x_3 \ldots x_n \geq x_{n+1} \geq \ldots$

Bounded Sequence ⇒

(i) Bounded above sequence :– A sequence $\{x_n\}$ is said to be bounded above if ~~their~~ $\exists$ (their exist) a real no. 'K' such that

$$x_n \leq K \quad \forall\ n \in N$$

(ii) Bounded below sequence :– A sequence $\{x_n\}$ is said to be bounded below if $\exists$ a real no. 'k' such that

$$x_n \geq k \quad \text{or} \quad k \leq x_n \quad \forall\ n \in N$$

(iii) ~~Only~~ Bounded sequence :– A sequence $\{x_n\}$ is said to be ~~only~~ bounded if $\exists$ a two real no. ~~'K'~~ ~~k~~ k and K, where $k \leq K$ such that

$$\boxed{k \leq x_n \leq K} \; ; \; \forall\ n \in N$$

(iv) Unbounded sequence :– If $\exists$ no real no. M such that

$$|x_n| \leq M \; ; \; \forall\ n \in N$$

then the sequence $\{x_n\}$ is said to be unbounded.

eg – $\{x_n\} = \frac{1}{n}$, $n \in N$

$= \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} - - - \}$

$0 \leq x_n \leq 1$

Given sequence $\{x_n\}$ is bounded & monotonically decreasing, $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} - -$

Limit of a sequence :- A sequence $\{x_n\}$ is said to approach the limit 'l' when $n \to \infty$, if for each $\epsilon > 0$, there exists ($\exists$) a positive m (depending upon ϵ) such that

$$|x_n - l| < \epsilon, \quad \forall \quad n \geq m$$

$\epsilon \to 0$

$$|x_n - l| = 0$$

$$x_n = l$$

$$\boxed{\lim_{n\to\infty} x_n = l}$$

Convergent, Divergent & Oscillating sequence →

Convergent Sequence :- A sequence $\{x_n\}$ is said to be convergent if

$$\boxed{\lim_{n\to\infty} \{x_n\} = \text{finite value}}$$

ex- Consider the sequence

$$\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3} - - - \frac{1}{2^n}$$

$$\{x_n\} = \frac{1}{2^n}$$

$$\lim_{n\to\infty} \{x_n\} = \lim_{n\to\infty} \frac{1}{2^n} = \frac{1}{2^\infty} = \frac{1}{\infty} = 0 = \text{finite}$$

Thus, $\{x_n\}$ is convergent.

(ii) Divergent Sequence ⇒ A sequence $\{x_n\}$ is said to be divergent if $\lim_{n\to\infty} \{x_n\} =$ Infinite value

$$\lim_{n\to\infty} \{x_n\} = -\infty \text{ or } \infty$$

eg - The sequence $\{n^2\}$

$$\{x_n\} = n^2$$

$$\lim_{n\to\infty} \{x_n\} = \lim_{n\to\infty} n^2 = \infty$$

$\{x_n\}$ is divergent.

(iii) Oscillatory Sequence :— If a sequence $\{x_n\}$ neither converge to a finite number nor diverger to $-\infty$ to $+\infty$ it is known as oscillatory sequence.

eg- Consider sequence

$$1-1+1-1+1-1+ - - - -$$

$$\{x_n\} = \begin{cases} 0 & , n \text{ is even} \\ 1 & , n \text{ is odd} \end{cases}$$

Q. Prove that the sequence $\langle x_n \rangle$ where $x_n = \frac{2n-7}{3n+2}$ is

(i) monotonically increasing
(ii) Bounded
(iii) Its limit is $2/3$

Ay- $$x_n = \frac{2n-7}{3n+2}, \quad \forall\, n \in N$$

$$x_{n+1} = \frac{2(n+1)-7}{3(n+1)-2} = \frac{2n-5}{3n+5}$$

$$x_{n+1} - x_n = \frac{2n-5}{3n+5} - \frac{2n-7}{3n+2}$$

$$= \frac{(2n-5)(3n+2) - (2n-7)(3n+5)}{(3n+5)(3n+2)}$$

$$= \frac{6n^2+4n-15n-10 - (6n^2+10n-21n-35)}{9n^2+6n+15n+10}$$

$$= \frac{\cancel{6n^2} - \cancel{11n} - 10 - \cancel{6n^2} + \cancel{11n} + 35}{9n^2+21n+10} = \frac{25}{(3n+2)(3n+5)}$$

$$= \frac{25}{(3n+2)(3n+5)} > 0$$

$$x_{n+1} - x_n > 0$$

$$x_{n+1} > x_n$$

So $\{x_n\}$ is monotonically increasing

(ii) $$x_n = \frac{2n-7}{3n+2} \quad \forall\, n \in N$$

for $n = 1, 2, 3, 4 \ldots$

$$x_1 = \frac{2(1)-7}{3(1)+2} = \frac{-5}{5} = -1$$

$$x_2 = \frac{2(2)-7}{3(2)+2} = \frac{-3}{8}$$

$$x_3 = \frac{2(3)-7}{3(3)+2} = \frac{-1}{11}$$

$$x_4 = \frac{2(4)-7}{3(4)+2} = \frac{1}{14}$$

and so on

, sequence $\left\{ -1, \frac{-3}{8}, \frac{-1}{11}, \ldots \right\}$

$\Rightarrow \quad -1 \leq x_n$ ✓

Again

$$1 - x_n = 1 - \frac{2n-7}{3n+2}$$

$$= \frac{3n+2-2n+7}{3n+2} = \frac{n+9}{3n+2} > 0$$

$$1 - x_n > 0$$

$$1 > x_n$$

so $\boxed{-1 \leq x_n < 1}$ $\{x_n\}$ is bounded

(iii) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{2n-7}{3n+2} = \lim\limits_{n\to\infty} \frac{2 - 7/n}{3 + 2/n}$

$$\boxed{\therefore \lim_{n\to\infty} x_n = \frac{2}{3}}$$

H.W.
Q. Prove the sequence $\langle x_n \rangle$ where $x_n = \frac{2n+1}{3n+5} \forall n \in N$

(i) monotonically increasing

(ii) bounded

(iii) $x_n = 2/3$

Ans-(i) $x_n = \frac{2n+1}{3n+5}$, $\forall\ n \in N$

$$x_{n+1} = \frac{2(n+1)+1}{3(n+1)+5} = \frac{2n+3}{3n+8}$$

$$x_{n+1} - x_n = \frac{2n+3}{3n+8} - \frac{2n+1}{3n+5}$$

$$= \frac{(3n+5)(2n+3) - (2n+1)(3n+8)}{(3n+8)(3n+5)}$$

$$= \frac{\cancel{6n^2} + 19n + 15 - \cancel{6n^2} - 11n - 8}{(3n+8)(3n+5)} = \frac{8n+7}{(3n+8)(3n+5)} > 0$$

So $\{x_n\}$ is monotonically increasing.

(ii) $x_n = \dfrac{2n+1}{3n+5}$, $(n = 1, 2, 3 - - -)$

$x_1 = \dfrac{2+1}{3+5} = \dfrac{3}{8}$, $x_2 = \dfrac{4+1}{6+5} = \dfrac{5}{11}$

$x_3 = \dfrac{6+1}{9+5} = \dfrac{7}{14} = \dfrac{1}{2}$, $x_4 = \dfrac{9}{17}$

and so on

Sequence $\{ 3/8, \frac{5}{11}, \frac{1}{2}, \frac{9}{17} - - - - \}$

Again

$1 - x_n = 1 - \dfrac{2n+1}{3n+5} = \dfrac{3n+5-2n-1}{3n+5}$

$= \dfrac{n+4}{3n+5} > 0$

$1 - x_n > 0$

$1 > x_n$

So $\boxed{\frac{3}{8} \leq x_n < 1}$ $\{x_n\}$ is bounded

(iii)

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \dfrac{2n+1}{3n+5}$

$\lim_{n\to\infty} x_n = \dfrac{n}{n} \lim_{n\to\infty} \dfrac{2+1/n}{3+5/n}$

$\lim_{n\to\infty} x_n = \dfrac{2+0}{3+0}$

$\Rightarrow$ $\boxed{\lim_{n\to\infty} x_n = \frac{2}{3}}$

Infinite Series ⇒ If $\{U_n\}$ be a sequence of real numbers, the sum of infinite no. of terms of this sequence.

i.e. $S_n = U_1 + U_2 + U_3 + \cdots + U_n$ then above series

$S_n = \sum_{n=1}^{\infty} U_n$ or $\sum U_n$ is know as infinite series.

$$\boxed{S_n = \sum U_n}$$

(i) Positive terms series:—

$\sum U_n = U_1 + U_2 + U_3 + \cdots + U_n + \cdots$

The series $\sum U_n$ is called positive terms series if each term of series is positive

(ii) Alternating Series ⇒

$\sum U_n = U_1 - U_2 + U_3 - U_4 + U_5 - \cdots$

★ IMP

(iii) Convergent series ⇒ A series $\sum U_n$ is said to be convergent if the sum of the first n terms of the tends to finite and unique, where $n \to \infty$

It is denoted by $\boxed{\lim_{n\to\infty} S_n = \text{finite and unique}}$

$$\boxed{\lim_{n\to\infty} \sum U_n = \text{finite and unique}}$$

(iv) divergent Series ⇒

$$\lim_{n\to\infty} S_n = -\infty \text{ to } \infty$$

(v) Oscillatory series ⇒

Oscillatory finitely	Oscillate infinitely
↓	↓
$\lim_{n\to\infty} S_n =$ finite but not unique	$\lim_{n\to\infty} S_n = \infty \text{ to } -\infty$

A necessary condition for convergence

Theorem:- A necessary condition for a positive term series ΣU_n to converges it that

$\boxed{\lim_{n\to\infty} U_n = 0}$ → Convergent

Remark:- If $\lim_{n\to\infty} U_n = 0$, we are not sure whether (चाहे) the series ΣU_n is convergent or not, but if $\lim_{n\to\infty} U_n \neq 0$ then the series ΣU_n is divergent.

Q. Test the convergence of the series $\sum U_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$

Ans- Here $U_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\infty} = 0$$

but

$$U_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}} = \frac{\sqrt{n+1}-\sqrt{n}}{\cancel{n}+1-\cancel{n}}$$

$$\lim_{n\to\infty} U_n = \sqrt{n+1} - \sqrt{n}$$

$$S_n = U_1 + U_2 + U_3 + \cdots + U_n + \cdots$$

$$S_n = (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots \sqrt{n+1} - \cancel{\sqrt{n}}$$

$$S_n = (\sqrt{n+1} - \sqrt{1})$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\sqrt{n+1} - 1) = \infty$$

$\sum U_n = S_n$ is divergent.

Cauchy's fundamental test for divergence $\Rightarrow$

If $\lim_{n\to\infty} U_n \neq 0$ then series $\sum U_n$ is divergence

Q- Test the convergence of the series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots \frac{n}{n+1} + \cdots \infty$$

Ans- We have $U_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n}{n+1} = \frac{\cancel{n}}{\cancel{n}} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$\left\{ \begin{array}{lll} r<1 & r^n \to 0 & , n \to \infty \\ r>1, & r^n \to \infty & , n \to \infty \end{array} \right\}$$

$\lim_{n\to\infty} U_n = 1 \neq 0$

Thus ΣU_n is divergence

Convergence of Geometric Series :-

The series $1 + r + r^2 + r^3 + \dots + r^{n-1} + \dots$

(i) Is convergent if $|r| < 1$
(ii) Is divergent if $r \geq 1$
(iii) Is oscillartory if $r \leq -1$

Solution $S_n = 1 + r + r^2 + \dots r^{n-1}$

$S_n = \dfrac{a(1-r^n)}{1-r}$ —— (1)

$\left\{ \begin{array}{l} a + ar + \dots ar^{n-1} \\ S_n = \dfrac{a(1-r^n)}{1-r} \end{array} \right\}$

(i) Where $|r| < 1$

$= \lim_{n\to\infty} r^n = 0$

from eq^n-(1)

$S_n = \dfrac{1-r^n}{1-r}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{1-r^n}{1-r} = \dfrac{1}{1-r}$ = finite

Hence the series is convergent

Positive term Series:-

Any series ΣU_n of all positive term is known a series of +ve term series if $U_n > 0$; $\forall n \in N$

Some important Test for convergence of infinite Series ⟹

$S_n = \Sigma U_n = \Sigma \frac{1}{n^p}$ $n=p$

(i) P - series test :-

The infinite series $\Sigma \frac{1}{n^p}$ i.e.

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \infty \text{ is}$$

(A) Convergent if $p > 1$

(B) Divergent , if $p \leq 1$

Q. Test the convergence of the series:-

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \infty$$

Ans- Given series can be written as

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$$

$$\Sigma U_n = \Sigma \frac{1}{\sqrt{n}} = \Sigma \frac{1}{n^{1/2}}$$

comparing with $\Sigma \frac{1}{n^p}$

$\Rightarrow p = 1/2 < 1$

By p- series test, ΣU_n is divergence.

(2) Comparison test :—

If ΣU_n and ΣV_n be two given series, such that

$$\lim_{n\to\infty} \frac{U_n}{V_n} = l \neq 0 \quad \text{(a finite non-zero quantity)}$$

than

ΣU_n & ΣV_n are either both convergent or both divergence.

example- Test the convergence of the series

$$\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots \infty$$

Solution- We have

$$\Sigma U_n = \frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \cdots + \frac{2n-1}{n(n+1)(n+2)} + \cdots$$

$$U_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{\cancel{n}(2-1/n)}{n^{\cancel{3}}(1+\frac{1}{n})(1+\frac{2}{n})}$$

$$= \lim_{n\to\infty} \frac{2-\frac{1}{n}}{n^2(1+\frac{1}{n})(1+\frac{2}{n})}$$

let auxillary series

$$V_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{(1+\frac{1}{n})(1+\frac{2}{n})} = 2$$

ΣU_n and Σv_n either both convergent or both divergence.

So Now $\Sigma v_n = \Sigma \frac{1}{n^2}$

$$\Sigma v_n = \Sigma \frac{1}{n^2} = \Sigma \frac{1}{n^p}$$

$$p = 2 > 1$$

$\Rightarrow$ Thus By p-series test, Σv_n i.e. ΣU_n is ~~divergence~~. convergent.

Q. Test the convergence of the series

$$\frac{1}{1} + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \dots$$

Ans - $\Sigma U_n = \frac{1}{2^2} + \frac{2^2}{3^3} + \dots \frac{(n-1)^{n-1}}{n^n} + \frac{n^n}{(n+1)^{n+1}}$

$$U_n = \frac{(n-1)^{n-1}}{n^n}$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(n)^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^n}}{n(1+\frac{1}{n})\cancel{n^n}(1+\frac{1}{n})^n}$$

$$= \lim_{n\to\infty} \frac{1}{n(1+\frac{1}{n})^{n+1}} \qquad \left\{ \because \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x \right\}$$

let auxillary series

$$V_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{U_n}{v_n} = \frac{\frac{1}{\cancel{n}(1+\frac{1}{n})(1+\frac{1}{n})^n}}{1/\cancel{n}} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})e}$$

$$= \lim_{n\to\infty} \frac{U_n}{V_n} = \frac{1}{e} \neq 0$$

POPU

Page No. :
Date :

By p-series test

$$\sum \frac{1}{n} = \sum \frac{1}{n^p} \Rightarrow p = 1$$

so thus, given $\sum U_n$ is divergent.

3.] D' Alemberts Ratio test :-

let $\sum U_n$ be a series of positive terms, such that

(I) if $\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = l$. then

(a) convergent if $l > 1$

(b) divergent if $l < 1$

(c) If $l = 1$ further investigation is ~~rea~~ needed.

(II) If $\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \infty$, then $\sum U_n$ is convergent.

Q. Test the convergent of the series

$$\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \cdots + \frac{(n+1)!}{3^n} + \cdots$$

Ans- we have

$$U_n = \frac{(n+1)!}{3^n}$$

So $U_{n+1} = \dfrac{(n+2)!}{3^{n+1}}$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{(n+1)!}{\cancel{3^n}} \times \frac{\cancel{3^n}.3^1}{(n+2)!}$$

$$= \frac{3\,\cancel{(n+1)!}}{(n+2)\cancel{(n+1)!}} = \lim_{n\to\infty} \frac{3}{n+2}$$

$$= \frac{3}{\infty} = 0 < 1$$

By D'Alembert ratio test ΣU_n is ~~then~~ divergent.

Q. Test the series:-

$$\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \frac{x^4}{4\cdot 5} + \cdots$$

Ans - $U_n = \frac{x^n}{(n)(n+1)}$

$$U_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{\cancel{x^n}}{n\cancel{(n+1)}} \times \frac{\cancel{(n+1)}(n+2)}{\cancel{x^n}\cdot x}$$

$$\lim_{n\to\infty} \frac{n+2}{n\cdot x} = \lim_{n\to\infty} \frac{\cancel{n}(1+2/n)}{\cancel{n}(x)}$$

$$= \frac{1}{x}$$

From D' Alembert's ratio test, we conclude that

(i) If $\frac{1}{x} > 1$ or $x < 1$ $\Rightarrow \Sigma U_n$ is convergent

(ii) If $\frac{1}{x} < 1$ or $x > 1$ $\Rightarrow \Sigma U_n$ is divergent

(iii) If $x = 1$, then ΣU_n becomes

$$U_n = \frac{1}{n(n+1)}$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n^2(1+1/n)}$$

let $V_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = \frac{1}{1} \neq 0$$

$$\Sigma U_n = \Sigma \frac{1}{n^2} = \Sigma \frac{1}{n^p} \Rightarrow p=2$$

$p > 1$ so convergent.

* Cauchy's root (or radical) test →

let ΣU_n be an infinite series of positive terms and let $\lim_{n\to\infty} [U_n]^{1/n} = l$

Then the series is

(i) Convergent if $l < 1$ and
(ii) Divergent if $l > 1$
(iii) If $l = 1$, then test fails.

Example- Test the convergence of $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

Sol- Here $U_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$$\lim_{n\to\infty} [U_n]^{1/n} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{-n^2}\right]^{1/n}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

By cauchey's root test, ΣU_n is convergent.

Q. Test the convergence of the series
$$\frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots \infty$$

Ay - ~~$\left(\frac{1}{2}\right)$~~ neglecting the first term

$$\left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots \left(\frac{n+2}{n+3}\right)^n x^n + \cdots$$

We have
$$U_n = \left(\frac{n+2}{n+3}\right)^n x^n \quad -(1)$$

$$\lim_{n\to\infty} [U_n]^{1/n} = \lim_{n\to\infty} \left(\left(\frac{n+2}{n+3}\right)^n x^n\right)^{1/n}$$

$$= \lim_{n\to\infty} \left(\frac{n+2}{n+3}\right) x$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+2/n)}{\cancel{n}(1+3/n)} x = x$$

By cauchey root test :—
if $x<1$ then convergent.
if $x>1$ then divergent
if $x=1$ then test fail.

from eq^n -(1) $\quad U_n = \left(\frac{n+2}{n+3}\right)^n$
for $x=1$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \left(\frac{n+2}{n+3}\right)^n = \lim_{n\to\infty} \frac{\cancel{n^n}(1+2/n)^n}{\cancel{n^n}(1+3/n)^n}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = \frac{e^2}{e^3} = \frac{1}{e} \neq 0$$

$\Rightarrow \Sigma U_n$ is divergent (By cauchey's fundamental test)

if $x<1$, ΣU_n is convergent
if $x>1$, ΣU_n is divergent.

Raabe's Test :–

Let ΣU_n be an infinite series of positive term and let

$$\lim_{n\to\infty}\left[n\left(\frac{U_n}{U_{n+1}}-1\right)\right]=l$$

Then the series is
(i) Convergent if $l>1$, and
(ii) Divergent if $l<1$,
(iii) If $l=1$, then further investigation is required or this test fails.

Q. Test the convergence of the series

$$1+\frac{3}{7}x+\frac{3\cdot6}{7\cdot10}x^2+\frac{3\cdot6\cdot9}{7\cdot10\cdot13}x^3+------$$

Ans – neglecting the first term

$$\left(\frac{3}{7}\right)x+\left(\frac{3\cdot6}{7\cdot10}\right)x^2+\left(\frac{3\cdot6\cdot9}{7\cdot10\cdot13}\right)x^3+-----$$

$$U_n=\frac{3\cdot6\cdot9\cdot12---3n}{7\cdot10\cdot13\cdot16---(3n+4)}x^n$$

$\{a_n=a+(n-1)d\}$ $$U_{n+1}=\frac{3\cdot6\cdot9\cdot12---(3n)(3n+3)}{7\cdot10\cdot13\cdot16---(3n+4)(3n+7)}x^{n+1}$$

$\therefore$ $$\frac{U_n}{U_{n+1}}=\frac{3n+7}{3n+3}\left(\frac{1}{x}\right) \quad \text{—— (1)}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty}\left(\frac{3+7/n}{3+3/n}\right)\left(\frac{1}{x}\right) = \left(\frac{\cancel{3}}{\cancel{3}}\right)\left(\frac{1}{x}\right)$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{1}{x}$$

from D' Alembert's ration test, we get

(i) if $\frac{1}{x} > 1$, or $x < 1$, ΣU_n is convergent

(ii) if $\frac{1}{x} < 1$ or $x > 1$, ΣU_n is divergent

(iii) if $x = 1$ then test fails

Now from (1), if we put $x = 1$

$$\frac{U_n}{U_{n+1}} = \frac{3n+7}{3n+3}$$

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{3n+7}{3n+3} - 1\right)$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{3n}+7-\cancel{3n}-3}{3n+3}\right]$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}}\left(\frac{4}{3+3/n}\right)$$

$$= \frac{4}{3} = l$$

By Raabe's test $l > 1$ $\quad 4/3 > 1$ so ΣU_n is convergent.

Q. Test the convergence of the series:–

(i) $1 + \frac{a}{1} + \frac{a(a+1)}{1\cdot 2} + \frac{a(a+1)(a+2)}{1\cdot 2\cdot 3}$

(ii) $1 + \frac{2}{3\cdot 4} + \frac{2\cdot 4}{3\cdot 5\cdot 6}$

Gauss test:-

If U_n is series of positive term such that

$$\frac{U_n}{U_{n+1}} = \alpha + \frac{\beta}{n} + \text{term of higher power}$$

$n, \alpha > 0$

(i) if $\alpha > 1$, then the series ΣU_n converges and if $\alpha < 1$, then the series ΣU_n is divergent

(ii) If $\alpha = 1$, then the series ΣU_n converges if $\beta > 1$ and divergence if $\beta \leq 1$

Q. Test the convergence of the series

$$1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2}$$

Ans- neglecting the first form,

$$\frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + - - -$$

$$U_n = \frac{2^2.4^2.6^2 8^2 - - - (2n)^2}{3^2.5^2.7^2.9^2 - - - (2n+1)^2}$$

$$U_{n+1} = \frac{2^2.4^2.6^2 - - - (2n)^2 (2n+2)^2}{3^2.5^2.7^2 - - (2n+1)^2 (2n+3)^2}$$

$$\frac{U_n}{U_{n+1}} = \frac{\cancel{(2n)^2}}{\cancel{(2n+1)^2}} \times \frac{\cancel{(2n+1)^2} (2n+3)^2}{\cancel{(2n)^2} (2n+2)^2}$$

$$\frac{U_n}{U_{n+1}} = \left(\frac{2n+3}{2n+2}\right)^2$$

POPU

Page No. :
Date :

$$\begin{cases} (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots \\ (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 - \dots \\ (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots \end{cases}$$

$$\cancel{\frac{U_n}{U_{n+1}} = \frac{1}{4}(2n+3)^2 (n+1)^{-2}}$$

$$\cancel{\frac{U_n}{U_{n+1}} = \frac{1}{4}[\ 4}$$

$$\frac{U_n}{U_{n+1}} = \frac{n^2}{n^2}\left(\frac{2+3/n}{2+2/n}\right)^2 = 1 \quad \text{~~test fails~~}$$

D'Alembert test fails

$$\cancel{\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\left(\frac{2n+3}{2n+2}\right)^2 - 1\right]}$$

$$\cancel{= \lim_{n\to\infty} n\left[\frac{(4n^2+12n+9) - (4n^2+8n+4)}{(2n+2)^2}\right]}$$

$$\cancel{= \lim_{n\to\infty} n\left[\frac{4n+5}{(2n+2)^2}\right] = \frac{n^2}{n^2}\left(\frac{4+5/n}{(2+2/n)^2}\right)}$$

$$\cancel{\Rightarrow 1} \qquad \text{Rabbe's test fails}$$

$$\cancel{\text{Again } \frac{U_n}{U_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2}}$$

$$\cancel{\log\left(\frac{U_n}{U_{n+1}}\right) = 2\log\left(\frac{2n+3}{2n+2}\right)} \Rightarrow 2\log(2n+3) - 2\log(2n+2)$$

$$\frac{U_n}{U_{n+1}} = \left(2+\frac{3}{n}\right)^2 \left(2+\frac{2}{n}\right)^{-2}$$

$$= \frac{1}{2^2}\left(4 + \frac{12}{n} + \frac{9}{n^2}\right)\left(1 - \frac{4}{2n} + \frac{24}{4n^2} - \dots\right)$$

$$\cancel{8} - \frac{12}{n} + \frac{12}{n} - \frac{48}{n^2} - \dots$$

$$= \frac{1}{4}\left(4 - \frac{16}{2n} \dots\right) + \left(\frac{12}{n} - \frac{48}{2n^2} + \dots\right) + \left(\frac{9}{n^2} + \dots\right)$$

$$= \frac{1}{4}\left[4 + \frac{24}{2n}\right) \qquad \alpha = 1,\ \beta = 1$$

ΣU_n divergent.

Logarithmic test :—

If ΣU_n is a series of positive term such that

$$\lim_{n\to\infty}\left[n \log\left(\frac{U_n}{U_{n+1}}\right)\right] = l$$

then (i) if $l > 1$, ΣU_n is convergent

(ii) if $l < 1$, ΣU_n is divergent

(iii) if $l = 1$, then further investigation is needed.

Leibnitz's test :—

If the alternating series $U_1 - U_2 + U_3 - U_4 \cdots$ $(U_n > 0)$ such that

(i) $U_{n+1} \le U_n$ $\forall\, n \in N$ and

(ii) $\lim_{n\to\infty} U_n = 0$, the series is convergent.

→ **Absolute convergence and Conditional convergence :—**

(i) **Absolute convergence series :—**

A series ΣU_n is said to be <u>absolutely convergent</u> of the positive term series $\Sigma |U_n|$ is convergent.

(ii) A series Σu_n is said to be conditionally convergent if the series $\Sigma |u_n|$ is divergent.
It is also known as semi convergent or exidentally convergent.

Q. Test the absolute convergence of the series
$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$$

Solution:-
$$\Sigma U_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$$
$$\Sigma |U_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \cdots$$
$$\Sigma |U_n| = \Sigma \frac{1}{2^{n-1}} = \Sigma V_n \quad (say)$$
$$V_n = \frac{1}{2^{n-1}}$$
$$V_{n+1} = \frac{1}{2^n}$$
$$\lim_{n\to\infty} \frac{V_n}{V_{n+1}} = \lim_{n\to\infty}\left(\frac{1}{2^{n-1}} \times 2^n\right) = \lim_{n\to\infty} 2^{n-n+1}$$
$$= 2 > 1$$
By ratio test, the series $\Sigma |U_n|$ is convergent
Since given series is alternating series and
~~By leibnitz's~~
$U_{n+1} \leq U_n$ (Continunally decreasing)
also
$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{2^{n-1}} = \frac{1}{2^\infty} = \frac{1}{\infty} = 0$$
By leibnitz's test ΣU_n is convergent.
Hence, given series is absolute convergence.

Q. Test the absolute convergence of the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - - - -$$

Ans- $\Sigma U_n = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + - - - -$

$$\Sigma |U_n| = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + - - - \frac{1}{\sqrt{n}} + - - -$$

$$\Sigma |U_n| = \Sigma \frac{1}{\sqrt{n}} = \Sigma V_n \text{ (say)}$$

$$V_n = \frac{1}{\sqrt{n}}$$

$$\cancel{V_{n+1} = \frac{1}{\sqrt{n+1}}}$$

Comparing with $\frac{1}{n^p}$

$$p = \frac{1}{2} < 1$$

By p-series test, $\Sigma V_n = \Sigma |U_n|$ is divergent

Since given series is alternating series and

Also $U_{n+1} \leq U_n$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n^{1/2}} = \frac{1}{\infty} = 0$$

By leibnitz's test ΣU_n is convergent.

Hence, given series is also conditional convergence.

Power Series :- A series of the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots a_n x^n = \sum_{n=0}^{\infty} a_n x^n$$

is said to be a power series. The a_n are called Coefficients of the power series.

~~Proof~~ :- Let $\sum U_n = \sum a_n x^n$

$\Rightarrow U_n = a_n x^n$

and $U_{n+1} = a_{n+1} x^{n+1}$

$$\therefore \lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty}\left(\frac{a_n x^n}{a_{n+1} x^{n+1}}\right) = \lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}} \cdot \frac{1}{x}\right) \quad (1)$$

if $\frac{a_n}{a_{n+1}} = \frac{1}{\lambda}$, then by D' Alembert ratio test

$$\lim_{n\to\infty}\left(\frac{1}{\lambda} \cdot \frac{1}{x}\right) = \frac{1}{\lambda x}$$

If $|\lambda x| < 1 \Rightarrow |x| < \frac{1}{\lambda}$ the series is convergent

Thus, the power series is convergent if $-\frac{1}{\lambda} < x < \frac{1}{\lambda}$

And outside of the interval the series is divergent.

Taylor's Series :- A taylor's series is a representation of functions as an infinite terms that are calculated at a single point.

$$\sum X_n = \sum_{n=0}^{\infty} \frac{f^n(a)}{\underline{|n}} (x-a)^n$$

$$= f^0(a) + f'(a)(x-a) + \frac{f''(a)}{\underline{|2}}(x-a)^2 + \cdots$$

where $f^n(a)$ denotes the n^{th} order derivative of f evaluated at a.

Some convergent series :—

(i) Exponential series
(ii) Binomial series
(iii) Logarithmic series
(iv) Trigonometric series

1.) Convergence of exponential Series :—

$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots \frac{x^n}{\lfloor n} + \cdots$ is convergent for all values of x.

Proof -

$$U_n = \frac{x^{n-1}}{\lfloor n-1}$$

$$U_{n+1} = \frac{x^n}{\lfloor n}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \left(\frac{x^{n-1}}{\lfloor n-1} \times \frac{\lfloor n}{x^n} \right) = \lim_{n\to\infty} \frac{n}{x}$$

By D' Alemberts test

If $\lim_{n\to\infty} \frac{n}{x} = \infty$, then by D'Alembert test ΣU_n is convergent.

2.) Convergence Logarithmic series :—

The series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{\lfloor n} + \cdots$$

is convergent for $-1 < x \le 1$.

Proof:- $$U_n = (-1)^{n-1}\frac{x^n}{\lfloor n}$$

$$U_{n+1} = (-1)^n \frac{x^{n+1}}{\lfloor n+1}$$

By ration test

$$\lim_{n\to\infty}\frac{U_n}{U_{n+1}} = \lim_{n\to\infty}(-1)^{n-1}\frac{x^n}{\lfloor n} \times \frac{\lfloor n+1}{x^{n+1}(-1)^n}$$

$$= \lim_{n\to\infty}(-1)\frac{(n+1)}{x\,n}$$

(By D'Almbert)

$$= \lim_{n\to\infty}(-1)\frac{\cancel{n}}{\cancel{n}}\left(\frac{1+1/n}{x}\right) = \frac{-1}{x}$$

Convergent $\left|\frac{1}{x}\right| > 1$ or $1 > |x|$

$$-1 > \frac{1}{x} > 1$$

$$-1 < x < 1$$

Divergent

$\left|\frac{1}{x}\right| < 1$ or $1 < |x|$

$$-1 > x > 1$$

At $x = 1$, then series

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots (-1)^{n-1}\frac{1}{n}$$

$$U_n > U_{n+1}$$

$$U_n = \frac{1}{n}$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty}\frac{1}{n} = 0$$

By leibnitz test, given series is convergent ~~divergent~~.

At $x=-1$, the series

$$= -1-\frac{1}{2}-\frac{1}{3}\cdots\cdots$$

$$= -\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\right)$$

$$\Sigma U_n = \sum(-1)\left(\frac{1}{n}\right)$$

$$\Sigma |U_n| = \Sigma \frac{1}{n} = \Sigma V_n \text{ (say)}$$

Comparing with $\Sigma \frac{1}{n^p}$

$p=1$

By p-series test $\Sigma V_n = \Sigma |U_n|$ is divergent

3.) Trigonometric test :-

$$\cos x = 1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}+\cdots (-1)^n \frac{x^{2n}}{\lfloor 2n}+\cdots\cdots$$

$$U_n = (-1)^n \frac{x^{2n}}{\lfloor 2n}$$

$$U_{n+1} = (-1)^{n+1} \frac{x^{2n+2}}{\lfloor 2n+2}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{(-1)^n x^{2n}}{\lfloor 2n} \times \frac{\lfloor 2n+2}{(-1)^{n+1} x^{2n+2}}$$

$$= \lim_{n\to\infty} \frac{-1}{x^2}(2n+2)(2n+1) = \infty$$

By D'Alembert, convergent.

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \infty$$

$$U_{n+1} < U_n$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} (-1)^n \frac{x^{2n}}{\lfloor 2n} = 0$$

POPU

Page No.:
Date: / /

By leibnitz series is Σu_n convergent.

(ii) $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots + (-1)^n \frac{x^{2n+1}}{\lfloor 2n+1}$

$$U_n = \frac{(-1)^n x^{2n+1}}{\lfloor 2n+1}$$

$$U_{n+1} = (-1)^{n+1} \frac{x^{2n+3}}{\lfloor 2n+3}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{(-1)^n x^{2n+1}}{\lfloor 2n+1} \times \frac{\lfloor 2n+3}{(-1)^{n+1} x^{2n+3}}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{(-1)(2n+3)(2n+2)}{x^2} = \infty$$

By d'Alemebert, convergent

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \infty$$

$$U_{n+1} < U_n$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(-1)^n x^{2n+1}}{\lfloor 2n+1} = 0$$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

POPU

Page No.:
Date:

Q. Test for convergence of the series

$$x + \frac{2^2 \cdot x^2}{\lfloor 2} + \frac{3^3 \cdot x^3}{\lfloor 3} + \frac{4^4 \cdot x^4}{\lfloor 4} + \cdots$$

Ans:- $x + \frac{2^2x^2}{\lfloor 2} + \frac{3^3x^3}{\lfloor 3} + \cdots + \frac{n^nx^n}{\lfloor n} + \cdots$

$$U_n = \frac{n^nx^n}{\lfloor n}$$

$$U_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{\lfloor n+1}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{n^nx^n}{\lfloor n} \times \frac{\lfloor n+1}{(n+1)^{n+1}\, x^{n+1}}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{\cancel{(n+1)}}{x} \cdot \frac{n^n}{(n+1)^{n+\cancel{1}}}$$

$$\lim_{n\to\infty} \left(\frac{1}{x}\right) \frac{\cancel{n^n}}{\cancel{n^n}\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty} \frac{1}{x\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}\cdot\frac{1}{x} = \frac{1}{ex}$$

By D'Alembert rule

if $\frac{1}{ex} < 1$ or $1 < ex$ or $\frac{1}{e} < x$

, then ΣU_n divergent

if $\frac{1}{ex} > 1$ or $\frac{1}{e} > x$, ΣU_n convergent

If $\frac{1}{ex} = 1 \Rightarrow x = \frac{1}{e}$, further investigation is required.

$$\lim_{n\to\infty}\left[n\log\left(\frac{U_n}{U_{n+1}}\right)\right]$$ To log series

$$\frac{U_n}{U_{n+1}} = \frac{n^n}{(n+1)^n}\cdot\frac{1}{x}$$

$$x=\frac{1}{e}, \quad \frac{U_n}{U_{n+1}} = \frac{e n^n}{(n+1)^n}$$

$$\lim_{n\to\infty} n\log\left(\frac{U_n}{U_{n+1}}\right) = \lim_{n\to\infty} n\log\left(\frac{e n^n}{(n+1)^n}\right)$$

$$\lim_{n\to\infty} n\log\left(\frac{e}{(1+\frac{1}{n})^n}\right) = \lim_{n\to\infty} n\log\left(\frac{e}{(1+\frac{1}{n})^n}\right)$$

$$\lim_{n\to\infty}\left[n\left\{1-n\log\left(1+\frac{1}{n}\right)\right\}\right]$$

$$\left[\because \log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-----\right]$$

$$\lim_{n\to\infty}\left[n\left\{1-n\left(\frac{1}{n}-\frac{(\frac{1}{n})^2}{2}+\frac{(\frac{1}{n})^3}{3}-----\right]\right.\right.$$

$$= \lim_{n\to\infty} n - \lim_{n\to\infty} n + \lim_{n\to\infty} \cancel{n^2}\times\frac{1}{\cancel{n^2}}\times\frac{1}{2} - 0 \ldots$$

$$= \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2} < 1$$

then ΣU_n is divergent.

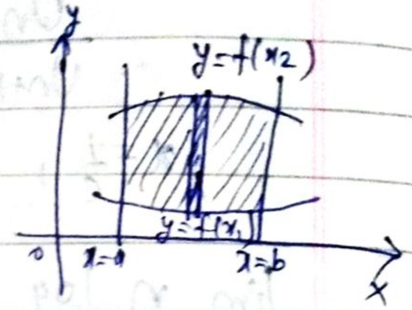
Unit - 5

Multiple Integration

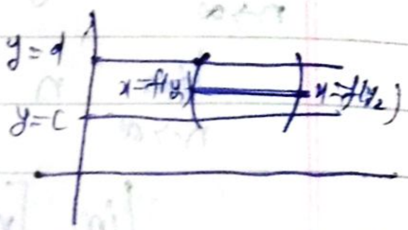
Cartiection

Double integration

$$\text{Area} = \int_{x=a}^{x=b} \int_{y=f(x_1)}^{y=f(x_2)} f(x,y)\, dx\, dy$$



$$\text{Area} = \int_{y=c}^{y=d} \int_{x=f(y_1)}^{x=f(y_2)} f(x,y) \cdot dy\, dx$$



Q.1 Evaluate $\iint xy\, dx\, dy$ over the region in positive quadrant for which $x+y \leq 1$

Ans we have

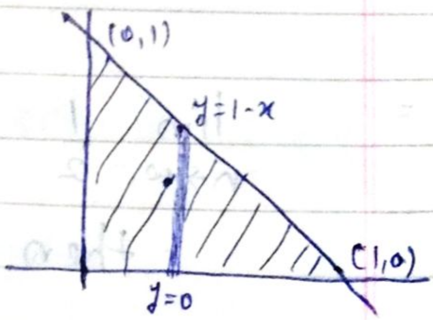
$x + y = 1$

$\frac{x}{1} + \frac{y}{1} = 1$ $\left[\frac{x}{a} + \frac{y}{b} = 1\right]$

$a = 1$, $b = 1$

$x = 0$ to $x = 1$

$y = 0$ to $y = 1 - x$



$$\text{Area} = \int_0^1 \int_0^{1-x} xy\, dx\, dy$$

$$= \int_0^1 x \left[\int_{y=0}^{1-x} y \cdot dy \right] \cdot dx$$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_0^{1-x} \cdot dx$$

$$\frac{1}{2}\int_0^1 x(1-x)^2 \cdot dx = \frac{1}{2}\int_0^1 x(1-2x+x^2)\cdot dx$$

$$\frac{1}{2}\int_0^1 x-2x^2+x^3 \cdot dx$$

$$\frac{1}{2}\left[\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]$$

$$\frac{1}{2}\left[\frac{1}{12}\right] = \frac{1}{24} \quad \text{Ans}$$

$\frac{1}{4}-\frac{8}{3}=\frac{9-8}{12}$

Q.2 Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$

Ans -

$$I = \int_{x=0}^1\left(\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\right)dx$$

$$I = \int_{x=0}^1\left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}\,dy\right]\cdot dx$$

$$I = \int_{x=0}^1\left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_{y=0}^{\sqrt{1+x^2}}\cdot dx$$

$$I = \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}}\tan^{-1}\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\cdot dx$$

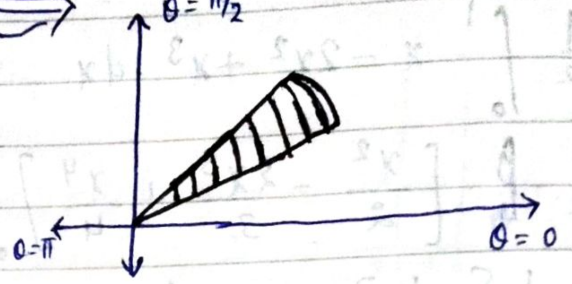
$$I = \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}}\tan^{-1}(1)\cdot dx$$

$$I = \frac{\pi}{4}\int_{x=0}^1 \frac{1}{\sqrt{1^2+x^2}}\cdot dx$$

$$I = \frac{\pi}{4}\left[\log\left[x+\sqrt{1+x^2}\right]\right]_0^1$$

$$I = \frac{\pi}{4}\,\log(1+\sqrt{2}) - \log(1) = \frac{\pi}{4}\log(1+\sqrt{2})$$

Evaluation of double integrals in Polar form ⟹



$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\theta = \tan^{-1}(y/x)$

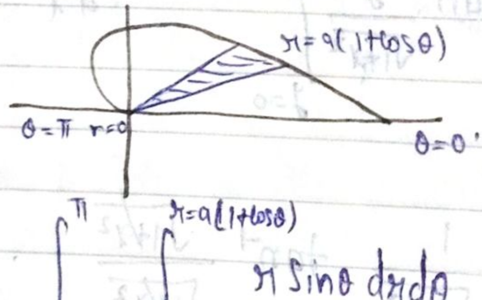
$dA = r\,d\theta\,dr = dx\,dy$

$$I = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta) \cdot dA$$

$$I = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta) \cdot r\,d\theta\,dr$$

Q.3 Find $\iint r\sin\theta\,dr\,d\theta$ over the area of the cardioide $r = a(1+\cos\theta)$ above the inital line

A -



$$A = \int_{\theta=0}^{\pi} \int_{r=0}^{r=a(1+\cos\theta)} r\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\int_0^{a(1+\cos\theta)} r \cdot dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \left(\frac{r^2}{2} \right)_0^{a(1+\cos\theta)} \cdot d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \sin\theta \; a^2 (1+\cos\theta)^2 \cdot d\theta$$

$$= \frac{r}{2}\int_0^{\pi} 2\sin\theta/_2 \cos\theta/_2 \,(a^2)\,(2\cos^2\theta/_2)^2 \cdot d\theta$$

$$= 4a^2 \int_0^{\pi} \sin\theta/_2 \cos^5\theta/_2 \cdot d\theta$$

let $\theta/_2 = t \Rightarrow \theta = 0,\ t = 0$

$dt = d\theta/_2 \Rightarrow \theta = \pi,\ t = \pi/_2$

$$A = 4a^2 \times 2 \int_0^{\pi/2} \sin t \cos^5 t \cdot dt$$

$\cos t = u$

$-\sin t \, dt = du$

$$A = -8a^2 \int_1^0 u^5 \cdot du$$

$$A = 8a^2 \left[\frac{u^6}{6}\right]_0^1 = \frac{8a^2}{6} = \frac{4a^2}{3}$$

Q.4 Evaluating the following integral by changing to polar co-ordinates

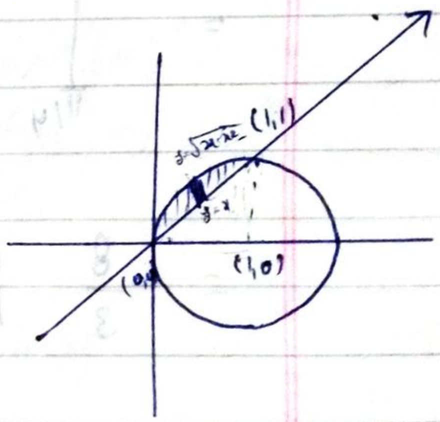
$$\int_0^1 \int_x^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\; dx\, dy$$

Ans- $x = 0$ to $x = 1$

$y = x$ —(1) to $y = \sqrt{2x - x^2}$

$y^2 = 2x - x^2$

$x^2 + y^2 - 2x = 0$ —(2)



using (1) in (2)

$2x^2 - 2x = 0$

$2x(x-1) = 0$

$x = 1,\ x = 0$

$y = x$ then $y = 1,\ y = 0$

$$\int_0^1 \left[\int_x^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\; dy\right] \cdot dx$$

Now changing in polar form

$x^2+y^2-2x=0$

$r^2 \quad -2r\cos\theta=0$

$r(r-2\cos\theta)=0$

$r=0$ to $r=2\cos\theta$

$\theta=\tan^{-1}(y/x)$

If $y=x$, $\quad \theta=\pi/4$

if $x=0$, $\quad \theta=\pi/2$

$$A=\int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{r=2\cos\theta} r.\,r d\theta \cdot dr$$

$$=\int_{\pi/4}^{\pi/2}\left[\int_{r=0}^{r=2\cos\theta} r^2 . dr\right].d\theta$$

$$=\int_{\pi/4}^{\pi/2}\left(\frac{r^3}{3}\right)_0^{2\cos\theta}.d\theta$$

$$=\frac{8}{3}\int_{\pi/4}^{\pi/2}\cos^3\theta \cdot d\theta$$

$$=\frac{8}{3}\int_{\pi/4}^{\pi/2}\frac{\cos3\theta+3\cos\theta}{4}.d\theta$$

$\cos3\theta=4\cos^3\theta-3\cos\theta$

$$\frac{2}{3}\left[\frac{\sin3\theta}{3}+3\sin\theta\right]_{\pi/4}^{\pi/2}$$

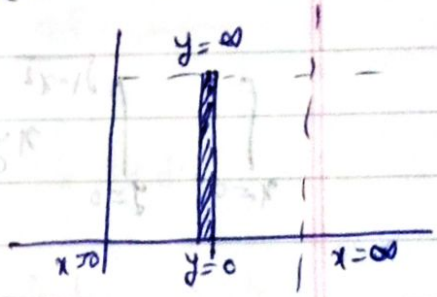
$$= \frac{2}{3}\left[-\frac{1}{3} + 3 - \left(\frac{+1}{3\sqrt{2}} + \frac{3}{\sqrt{2}}\right)\right]$$

$$\frac{2}{3}\left[\frac{-\sqrt{2} + 9\sqrt{2} - 1 - 9}{3\sqrt{2}}\right] = \frac{2}{3}\left[\frac{8\sqrt{2}}{3\sqrt{2}} - \frac{10}{3\sqrt{2}}\right]$$

$$= \frac{2}{3}\left[\frac{8}{3} - \frac{10}{3\sqrt{2}}\right] = \cancel{\frac{20}{9} - \frac{16}{9\sqrt{2}}}$$

$$\frac{16}{9} - \frac{20}{9\sqrt{2}} \quad \underline{\text{Ans}}$$

Q. Evaluate the integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar cordinate.



Ans - let $x = r\cos\theta$, $y = r\sin\theta$

$r^2 = x^2 + y^2$

$x = 0,\ y = 0 \Rightarrow r = 0$

$x = \infty,\ y = \infty \rightarrow r = \infty$

$\theta = \tan^{-1}(y/x)$

if $y = 0$ $\quad \theta = \tan^{-1}(0/x) = \tan^{-1}(0) = 0$

if $x = 0$ $\quad \theta = \tan^{-1}(y/0) = \tan^{-1}(\infty) = \pi/2$

then $\begin{cases} \theta = 0 \text{ to } \theta = \pi/2 \\ r = 0 \text{ to } r = \infty \end{cases}$

$$\int_0^{\pi/2}\int_0^\infty e^{-r^2} \cdot r\, d\theta\, dr$$

$$\int_0^{\pi/2}\left[\int_0^\infty e^{-r^2} \cdot r\, dr\right] \cdot d\theta$$

$r^2 = t$

$r\,dr = dt/2$

$$\int_0^{\pi/2}\left[\frac{1}{2}\int_0^\infty e^{-t} \cdot dt\right] \cdot d\theta$$

$$-\int_0^{\pi/2}\left[\frac{1}{2}\left(e^{-\infty}-e^{0}\right)\cdot d\theta\right.$$

$$= +\frac{\pi}{4}$$

Triple Integral :-

Q. Evaluate the integral

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\ dx\,dy\,dz$$

Ans -

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\left[\int_{z=0}^{\sqrt{1-x^2-y^2}} z\ dz\right] dx\,dy$$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\,(1-x^2-y^2)\cdot dx\,dy$$

$$\frac{1}{2}\int_{x=0}^{1} dx\int_{y=0}^{\sqrt{1-x^2}}\left(xy-x^3y-xy^3\right)\cdot dy$$

$$\frac{1}{2}\int_{x=0}^{1}\left[\frac{xy^2}{2}-\frac{x^3y^2}{2}-\frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$\frac{1}{2}\int_{x=0}^{1}\frac{x(1-x^2)}{2}-\frac{x^3(1-x^2)}{2}-\frac{x(1-x^2)^2}{4}\cdot dx$$

$$\frac{1}{4}\int_{x=0}^{1} x - x^3 - x^3 + x^5 - \left(\frac{x + 2x^3 - x^5}{2}\right) \cdot dx$$

$$\frac{1}{8}\int_{x=0}^{1} \left(2x - x - \cancel{2x^3} + \cancel{2x^3} - 2x^3 + 2x^5 - x^5\right) \cdot dx$$

$$\frac{1}{8}\int_{x=0}^{1} \left(x - 2x^3 + x^5\right) \cdot dx$$

$$\frac{1}{8} \left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6}\right]_0^1$$

$$\frac{1}{8}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = \frac{1}{48} \quad \underline{\text{Ay}}$$

Q. Evaluate $\int_0^a \int_0^x \int_0^{y+x} e^{x+y+z} \, dx\,dy\,dz$

Ay –
$$\int_{x=0}^{a}\int_{y=0}^{y=x}\int_{z=0}^{z=y+x} e^{x+y+z} \cdot dx\,dy\,dz$$

$$\int_{x=0}^{a}\int_{y=0}^{y=x}\int_{z=0}^{z=y+x} e^{x} \cdot e^{y} \cdot e^{z} \, dx\,dy\,dz$$

$$\int_{x=0}^{a}\int_{y=0}^{y=x} e^{x+y}\left[\int_{z=0}^{z=y+x} e^{z} \cdot dz\right] dx\,dy$$

$$\int_{x=0}^{a}\int_{y=0}^{y=x} e^{x+y}\left(e^{z}\right)_0^{y+x} \cdot dx\,dy$$

$$\int_{x=0}^{a}\int_{y=0}^{y=x} e^{x+y}\left(e^{x+y} - 1\right) \cdot dx\,dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{y=x} \left[e^{2(x+y)} - e^{(x+y)}\right] . dx\, dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{y=x} e^{2x} . e^{2y} - e^{x} . e^{y} . dx\, dy$$

$$= \int_{x=0}^{a} \left[\int_{y=0}^{y=x} \left(e^{2x} . e^{2y} - e^{x} . e^{y}\right) . dy \right] dx$$

$$= \int_{x=0}^{a} \left[\left[e^{2x} . \frac{e^{2y}}{2} - e^{x} . e^{y} \right]_{0}^{y=x} . dx \right.$$

$$= \int_{x=0}^{a} \frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{} + e^{x} . dx$$

$$\left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x} \right]_{0}^{a}$$

$$\cancel{\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} - \frac{3}{2}a - \frac{3}{8}}$$

$\frac{1}{8} - \frac{1}{2} = \frac{1-4}{8}$

$$\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{3}{8} \quad \text{Ans}$$

CHANGE OF ORDER :-

$$\int_{x=a}^{b} \int_{y=f(x_1)}^{y=f(x_2)} f(x,y)\, dx\, dy = \int_{y=g(x_1)}^{y=g(x_2)} \int_{x=c}^{x=d} f(x,y)\, dy\, dx$$

Q.1 Change the order of integration in the following integrals

(i) $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} f(x,y)\,dx\,dy$

(ii) $\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$

~~(iii)~~ $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dx\,dy$

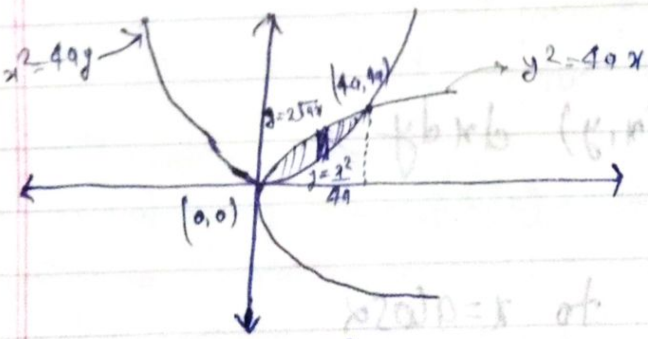
(iv) $\int_0^1\int_{e^x}^{e} \frac{1}{\log y}\,dx\,dy$

Ans-(i) $\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} f(x,y)\,dx\,dy$

Limit $x=0$ to $x=4a$

$y=\frac{x^2}{4a}$ to $y=2\sqrt{ax}$

$x^2=4ay$ —① and $y^2=4ax$ —②



$x^2=4ay$

$x^4=16a^2y^2$

(using(2)) $x^4=16a^2\times 4ax$

$x^4 - 64a^3x = 0$

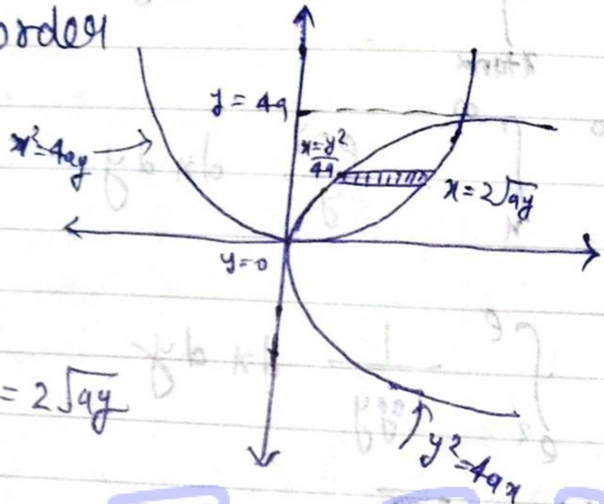
$x(x^3 - 64a^3) = 0$

$x = 0 \quad , \quad x = 4a$

$y = 0 \quad , \quad y = 4a$

intersection point $(0,0)$ & $(4a, 4a)$

change of order



$y = 0$ to $y = 4a$

$x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$

$$\int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} f(x,y) \cdot dx\, dy$$

$$\int_{y=0}^{4a} \int_{x=y^2/4a}^{x=2\sqrt{ay}} f(x,y) \; dy\, dx$$

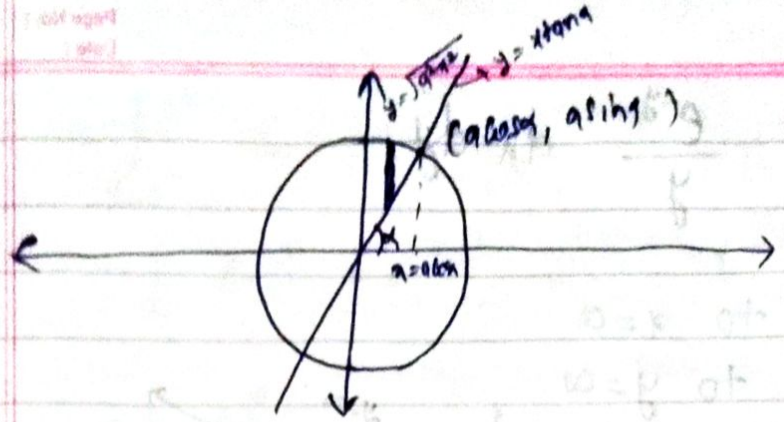
(ii) $$\int_{0}^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y) \; dx\, dy$$

limit $x = 0$ to $x = a\cos\alpha$

$y = x\tan\alpha$ —(1) to $y = \sqrt{a^2 - x^2}$ —(2)

$\tan^{-1}\left(\frac{y}{x}\right) = \alpha$ & $x^2 + y^2 = a^2$

is eqn of line.



$y = x\tan\alpha$

$y^2 = x^2\tan^2\alpha$

Using (c) $y^2 = a^2 + x^2$ (eqⁿ of circle)

$x^2\tan^2\alpha + x^2 = a^2$

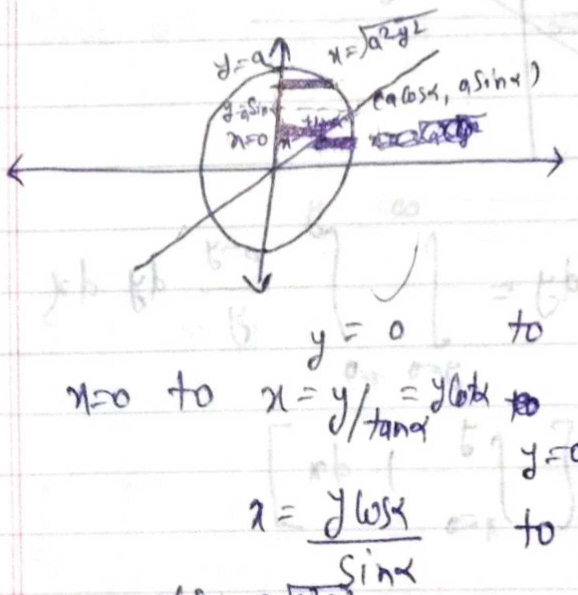
$x^2(1+\tan^2\alpha) = a^2$

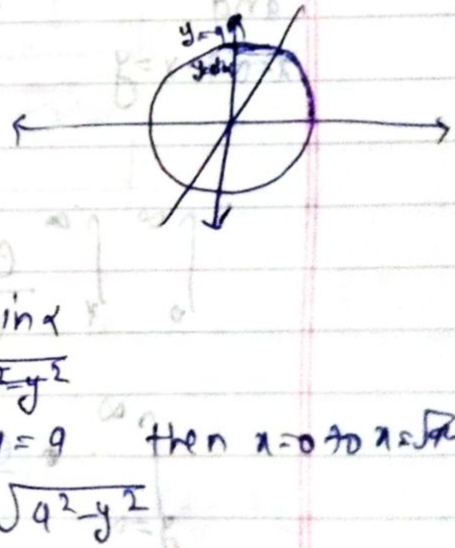
$x^2 = a^2\cos^2\alpha$

$x = a\cos\alpha$

$y = a\sin\alpha$

intersection point $(0,0)$ $(a\cos\alpha, a\sin\alpha)$





$y = 0$ to $y = a\sin\alpha$

$x=0$ to $x = y/\tan\alpha = y\cot\alpha$ ~~to~~ $\cancel{x = \sqrt{a^2-y^2}}$

$y = a\sin\alpha$ to $y = a$ then $x = 0$ to $x = \sqrt{a^2-y^2}$

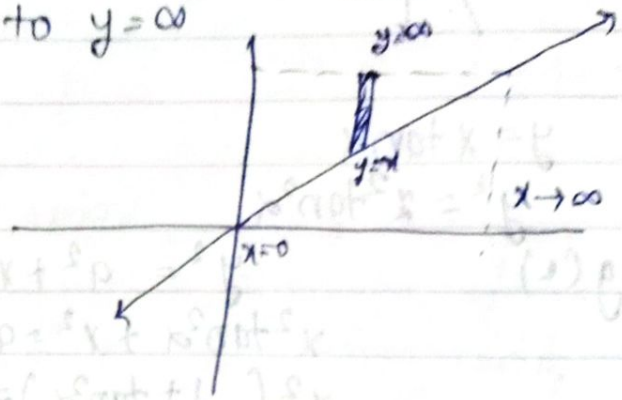
$x = \dfrac{y\cos\alpha}{\sin\alpha}$ to $x = \sqrt{a^2-y^2}$

$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$$

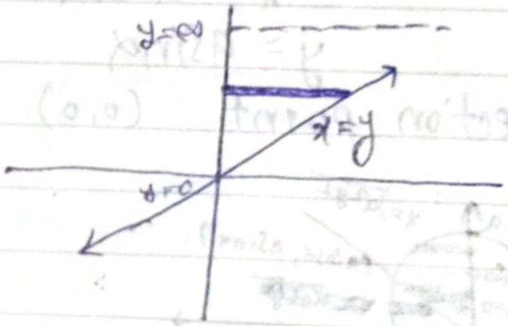
$$\int_0^{a\sin\alpha} = \int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x,y)\,dy\,dx + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{a^2-y^2}} f(x,y)\,dy\,dx$$

(iii) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$

Ans- $x=0$ to $x=\infty$
$y=x$ to $y=\infty$



Now on changing order of integration, we have limits
$y=0$ to $y=\infty$
and
$x=0$ to $x=y$



$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\, dy\, dx$$

$$\int_{y=0}^{\infty} \frac{e^{-y}}{y}\, dy \left[\int_{x=0}^{y} 1 \cdot dx \right]$$

$$\int_{y=0}^{\infty} \frac{e^{-y}}{y}\, dy\, [x]_0^y$$

$$\int_{y=0}^{\infty} \frac{e^{-y}}{\cancel{y}} \cdot \cancel{y} \cdot dy$$

$$\int_{y=0}^{\infty} e^{-y}\,.dy = -\left[e^{-y}\right]_0^{\infty}$$

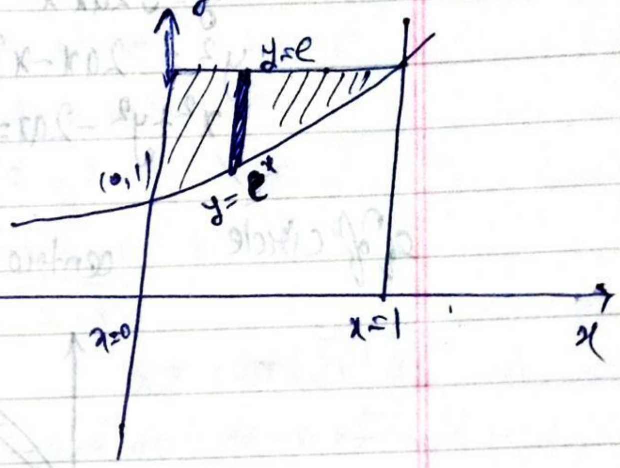
$$-\left[e^{-\infty} - e^{0}\right] = 1$$

(iv) $\int_0^1 \int_{e^x}^{e} \frac{1}{\log y}\,dx\,dy$

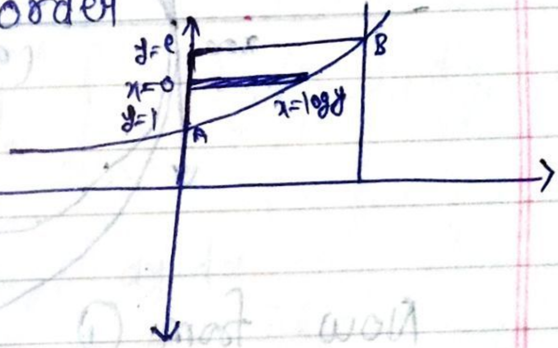
Ay – $x=0$ to $x=1$

$y=e^x$ to $y=e$

$2<e<3$



Now on changing order



$y=1$ to $y=e$

$x=0$ to $x=\log y$

$$\int_0^1 \int_{e^x}^{e} \frac{1}{\log y}\,dx\,dy = \int_{y=1}^{e} \int_{x=0}^{\log y} \frac{1}{\log y}\,dy\,dx$$

$$\int_{y=1}^{e} \left[\frac{1}{\log y} \int_{x=0}^{\log y} 1 . dx\right] dy$$

$$\int_{y=1}^{e} \frac{1}{\cancel{\log y}} \times \cancel{\log y}\ .dy = e-1 \text{ Ans}$$

Q. Change the order of integration of the following integral:—

$$\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\, dx\, dy$$

Ans- Limits

$x=0$ to $x=2a$

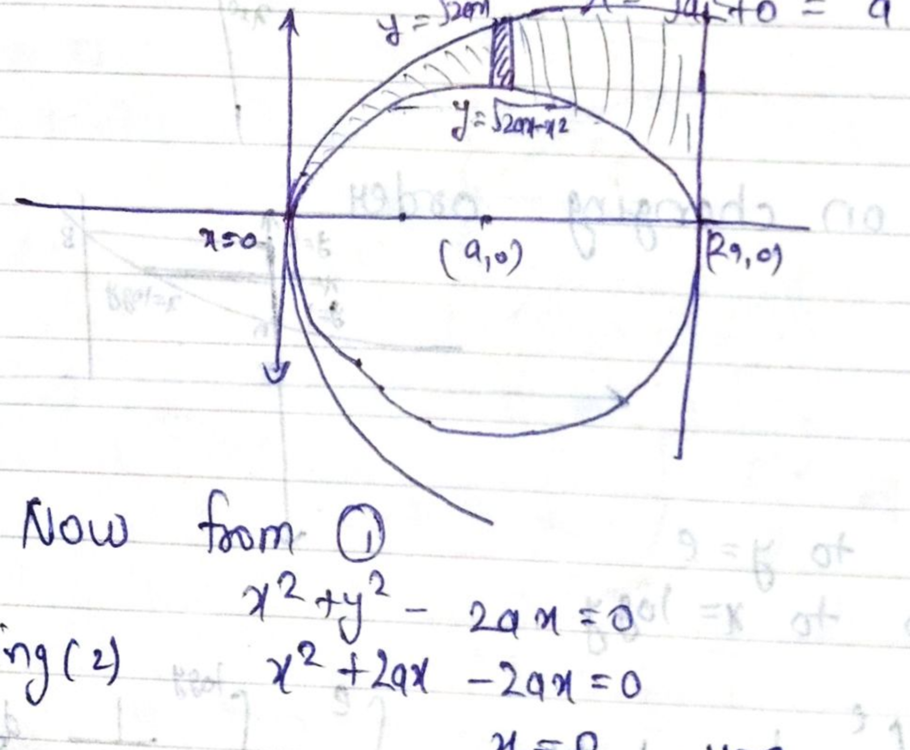
$y=\sqrt{2ax-x^2}$ to $y=\sqrt{2ax}$

$y^2 = 2ax-x^2$ to $y^2 = 2ax$

$x^2+y^2-2ax=0$ — (1) to $y^2 = 2ax$ — (2)

eq of circle centre $(-g,-f) = (a,0)$

$r = \sqrt{a^2+0} = a$



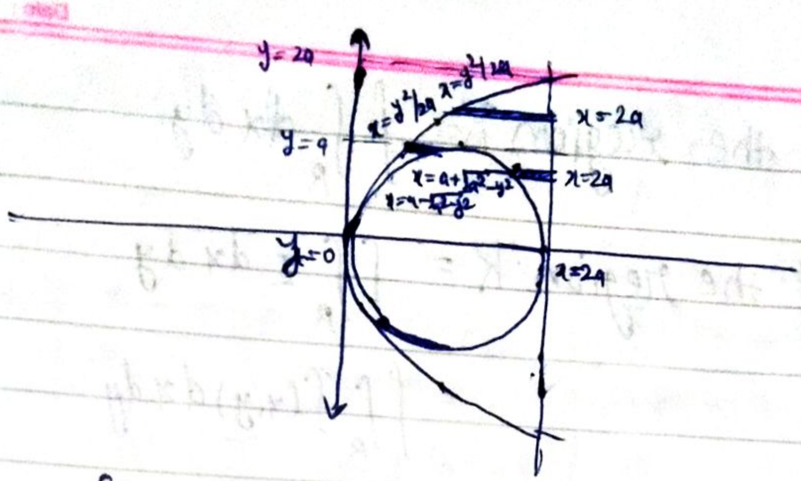
Now from (1)

$x^2+y^2-2ax=0$

using (2) $x^2+2ax-2ax=0$

$x=0,\ y=0$

Now on changing order of integration we have



from eqⁿ

$$x^2 + y^2 - 2ax = 0$$

$$x^2 - 2ax + a^2 \quad + y^2 - a^2 = 0$$

$$(x-a)^2 + y^2 = a^2$$

$$(x-a)^2 = a^2 - y^2$$

$$x = \pm\sqrt{a^2-y^2} + a$$

$$x = a \pm \sqrt{a^2-y^2}$$

$x = a - \sqrt{a^2-y^2}$, $x = a + \sqrt{a^2-y^2}$

~~for 1st quardric $x = a + \sqrt{a^2-y^2}$~~

$$\int_0^a \int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x,y)\, dy\, dx + \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\, dy\, dx +$$

$$\int_a^{2a} \int_{y^2/2a}^{2a} f(x,y)\, dy\, dx$$

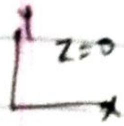
Applications of double integrals :—

(i.) Surface area
(ii) Volume
(iii) Mass
(iv) Centre Gravity (CG)

(i) Area of the region $R = \iint_R dx\,dy$

(ii) Volume of the region $R = \iint_R z\,dx\,dy$

$$= \iint_R f(x,y)\,dx\,dy$$



Q.1 find by double integration the area of the regions bounded by following curves

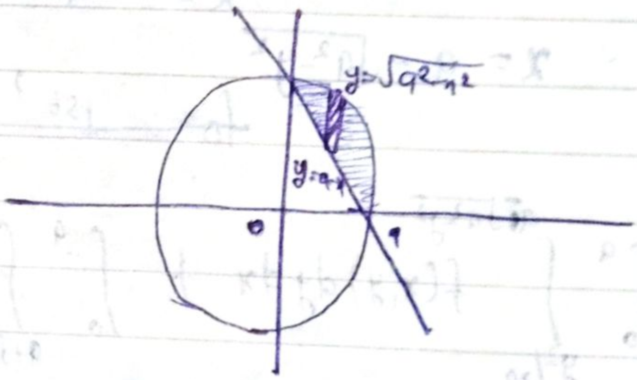
(i) $x^2+y^2=a^2$ and $x+y=a$ (in the 1st quardent)

(ii) $(x^2+4a^2)y = 8a^3$, line $y=\frac{1}{2}x$ & y-axis

Ans – we have

(i) $x^2+y^2=a^2$

centre (0,0)



line $x+y=a$

$\frac{x}{a}+\frac{y}{a}=1$

$x=0$ to $x=a$

$y=a-x$ to $y=\sqrt{a^2-x^2}$

Required area $= \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dx\,dy$

$$= \int_{x=0}^{a} \left[\int_{y=a-x}^{\sqrt{a^2-x^2}} 1 \cdot dy \right] dx$$

$$= \int_{x=0}^{a} \sqrt{a^2-x^2} - a + x \cdot dx$$

$$= \int_{x=0}^{a} \sqrt{a^2-x^2}\,.dx - \int_{x=0}^{a} (a-x)\,.dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a - \left(ax - \frac{x^2}{2}\right)_0^a$$

$$= \frac{a^2}{2} \times \frac{\pi}{2} - \left(\frac{2a^2-a^2}{2}\right)$$

$$= \frac{a^2}{2}\left(\frac{\pi-2}{2}\right)$$

(ii) We have

$$(x^2+4a^2)\,y = 8a^3 \quad \text{—— (1)}$$

$$y = x/2 \quad \text{—— (2)}$$

At y-axis

$$\Rightarrow \quad x = 0$$

from (1) $\quad (x^2+4a^2)\,y = 8a^3$

$$y = \frac{8a^3}{x^2+4a^2} \quad \text{—— (3)}$$

from (1) $\quad (x^2+4a^2)y = 8a^3$

Using (2) $\quad (x^2+4a^2)\left(\frac{x}{2}\right) = 8a^3$

$$x^3 + 4a^2x = 16a^3$$

$$x^3 + 4a^2x - 16a^3 = 0$$

$x = 2a$, $\quad 8a^3 + 8a^3 - 16a^3 = 0 \Rightarrow 16a^3 - 16a^3 = 0$

limit $\quad x=0$ to $x=2a$

$y = x/2$ to $y = \dfrac{8a^3}{x^2+4a^2}$

Required area $= \iint_R dx\,dy$

$$= \int_0^{2a} \int_{x/2}^{\frac{8a^3}{x^2+4a^2}} 1 \,.\, dx\,dy$$

POPU

Page No.:
Date:

$$\int_0^{2a}\left[\int_{y=a/2}^{y=\frac{8a^3}{x^2+4a^2}} 1.dy\right].dx$$

$$\int_0^{2a}\left(\frac{8a^3}{x^2+4a^2}-\frac{x}{2}\right).dx$$

$$8a^3\int_0^{2a}\frac{1}{x^2+(2a)^2}.dx - \int_0^{2a}\frac{x}{2}.dx$$

$$8a^3\left[\frac{1}{2a}\tan^{-1}\left(\frac{x}{2a}\right)\right]_0^{2a} - \left[\frac{x^2}{4}\right]_0^{2a}$$

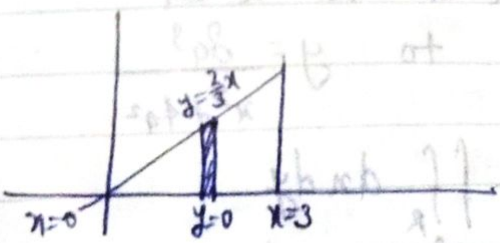
$$8a^3\left[\frac{1}{2a}\tan^{-1}\left(\frac{2a}{2a}\right)-\frac{1}{2a}\tan^{-1}(0)\right]-\left[\frac{(2a)^2}{4}-0\right]$$

$$\frac{\overset{2}{\cancel{8}}a^{\cancel{3}}}{\cancel{2a}}\times\frac{\pi}{\cancel{4}} - \frac{\cancel{4}a^2}{\cancel{4}}$$

$$= a^2(\pi-1) \quad \underline{\text{Ay}}$$

Q.2 Find the volume under the plane $x+y+z=6$ & above the triangle in xy plane bounded by $2x=3y$, $y=0$, $x=3$

Ay - volume $= \iint_R z\, dx\, dy$



plane

$x+y+z=6$

$z=6-x-y$

POPU

Page No. :

Date : / /

$$\int_0^3 \int_0^{\frac{2}{3}x} (6-x-y)\, dx\, dy$$

$$\int_0^3 \left[\int_0^{2/3x} (6-x-y)\, dy\right] dx$$

$$\int_0^3 \left[6y - xy - y^2/_2\right]_0^{2/3x} . dx$$

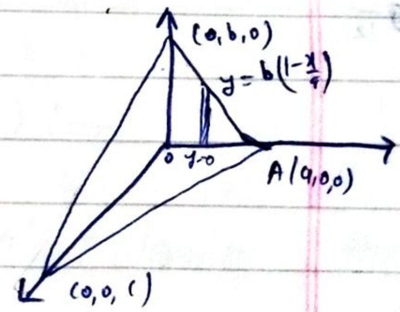
$$\int_0^3 4x - \frac{2}{3}x^2 - \frac{2x^2}{y} . dx$$

$$\left[\frac{\cancel{4}^{2}x^2}{\cancel{2}} - \frac{2x^3}{9} - \frac{2x^3}{27}\right]_0^3$$

$$18 - 2\times3 - 2$$

$$18 - 6 - 2 = 10 \ \underline{Ay}$$

Q.3 Find the volume bounded by the cordinate ~~planes~~ axis & plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$



Ay –

we have

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \quad —①$$

On XY Plane $z = 0$

$x = 0$ to $x = a$

$y = 0$ to $y = b\left(1 - \frac{x}{a}\right)$

$$\text{Volume} = \int_0^a \left[\int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\right] . dx$$

$$\int_0^a \left[cy - \frac{xyc}{a} - \frac{y^2 c}{2b} \right]_0^{b\left(\frac{a-x}{a}\right)} dx$$

$$\int_0^a \left[\frac{bc(a-x)}{a} - \frac{x\,b(a-x)c}{a^2} - \frac{b^2(a-x)^2 c}{2a^2 b} \right] dx$$

$$\left[\frac{bcax}{a} - \frac{bcx^2}{2a} - \frac{x^2 bac}{2a^2} + \frac{cbx^3}{3a^2} - \frac{cb}{2ab}\left(a^2x - \frac{2ax^2}{2} + \frac{x^3}{3}\right) \right]_0^a$$

$$abc - \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{3} - \frac{bc}{2}\left(a^3 - a^3 + \frac{a^3}{3}\right)$$

$$\cancel{\frac{abc}{2} - \frac{5abc}{6} - \frac{a^3bc}{6}}$$

$$\cancel{bc\left[\frac{ac}{2}\right]} \quad \cancel{bc\left[\frac{c}{12} - \frac{a^3}{6}\right]}$$

$$+\frac{abc}{3} - \frac{b^2c}{2a^2b}\left(a^3 - a^3 + \frac{a^3}{3}\right)$$

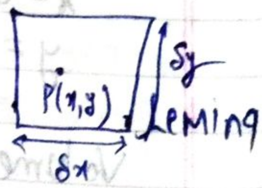
$$+\frac{abc}{3} - \frac{abc}{6}$$

$$\frac{-abc + 2abc}{6} =$$

$$\frac{abc}{6}\ u^3$$

Calculation of Mass :-

(a) For a plane leming (solid) :-



let the surface density at the point P be $f = f(x,y)$

$\therefore$ Elementary mass at $P = \rho\, \delta x\, \delta y$

$$\boxed{\text{Therefore total mass of the lemina} = \iint_R \rho \, dx\, dy}$$

- If given curve in polar form $\rho = f(r,\theta)$

Total mass of the lemina $= \iint \rho\, r\, d\theta\, dr$

(b) For a solid, let the density at the point be $\rho = f(x,y,z)$

$\therefore$ Elementory mass at $P = \rho\, \delta x\, \delta y\, \delta z$

$$\boxed{\text{Total mass of the solid} = \iiint \rho\, dx\, dy\, dz}$$

Centre of Gravity (C.G.) :-

(a) To find C.G. $(\bar{x}, \bar{y})$ of a plane lemina, then

$$\bar{x} = \frac{\iint x \rho\, dx\, dy}{\iint \rho\, dx\, dy}, \quad \bar{y} = \frac{\iint y \rho\, dx\, dy}{\iint \rho\, dx\, dy}$$

(b) For polar co-ordinates

$$\bar{x} = \frac{\iint r\cos\theta \cdot \rho\, r\, dr\, d\theta}{\iint \rho\, r\, dr\, d\theta}, \quad \bar{y} = \frac{\iint r\sin\theta\, \rho\, r\, d\theta\, dr}{\iint \rho\, r\, d\theta\, dr}$$

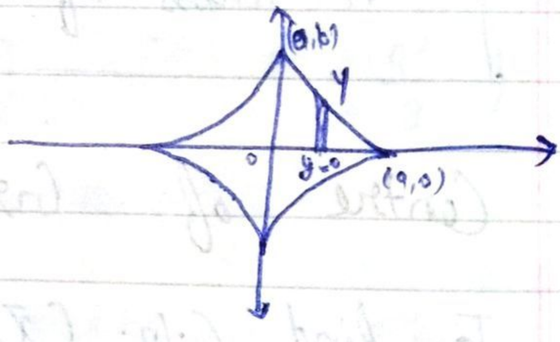
(c) To find C.G. $(\bar{x}, \bar{y})$ of solid, we have

$$\bar{x} = \frac{\iiint x \rho \, dx\,dy\,dz}{\iiint \rho \, dx\,dy\,dz}, \quad \bar{y} = \frac{\iiint y \rho \, dx\,dy\,dz}{\iiint \rho \, dx\,dy\,dz}$$

$$\bar{z} = \frac{\iiint z \rho \, dx\,dy\,dz}{\iiint \rho \, dx\,dy\,dz}$$

Q. Use double integration the find C.G. of a lemina in the shape of a quadrant of the $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ the density being $\rho = kxy$, where $k \to$ con.

Ans. $\bar{x} = \dfrac{\iint x \rho \, dx\,dy}{\iint \rho \, dx\,dy}$



$x = 0$ to $x = a$
$y = 0$ to $y = y$

$$\bar{x} = \frac{\int_0^a \int_0^y x(kxy)\,dx\,dy}{\int_0^a \int_0^y (kxy)\,dx\,dy}$$

$$\bar{y} = \frac{\int_0^a \int_0^y y(kxy)\,dx\,dy}{\int_0^a \int_0^y (kxy)\,dx\,dy}$$

POPU

Page No.
Date

$$\bar{x} = \frac{\int_0^a \left[\int_0^y (x^2 k) y \, dy \right] dx}{\int_0^a (kx) \left[\int_0^y y \cdot dy \right] \cdot dx}$$

$$\bar{x} = \frac{\int_0^a kx^2 \left[\int_0^y y \cdot dy \right] dx}{\int_0^a kx \left[\int_0^y y \cdot dy \right] dx} = \frac{\cancel{\frac{1}{2}} \int_0^a x^2 \cdot y^2 \cdot dx}{\cancel{\frac{y}{2}} \int_0^a y^2 \cdot dx}$$

$$\bar{x} = \frac{y^2 \int_0^a x^2 dx}{y^2 \int_0^a x \, dx} \qquad \bar{x} = \frac{\int_0^a x^2 y^2 dx}{\int_0^a x y^2 dx}$$

$\because \quad x = a\cos^3\theta , \quad y = b\sin^3\theta$

$dx = -3a\cos^2\theta \sin\theta \cdot d\theta, \; dy = 3b\sin^2\theta \cos\theta \cdot d\theta$

$$\bar{x} = \frac{\int_{\pi/2}^0 a^2\cos^6\theta \, b^2\sin^6\theta \left(-3a\cos^2\theta \sin\theta \right) \cdot d\theta}{\int_{\pi/2}^0 (a\cos^3\theta)(b^2\sin^6\theta) \cdot (-3a\cos^2\theta \sin\theta) d\theta}$$

$$\bar{x} = \frac{-3\int_{\pi/2}^0 a^3 \cos^8\theta \sin^7\theta \cdot d\theta}{-3a^2b^2 \int_{\pi/2}^0 \sin^7\theta \cdot \cos^5\theta \, d\theta}$$

$$\bar{x} = \frac{-3a^3b^2}{-3a^2b^2} \frac{\int_{\pi/2}^0 \cos^8\theta \sin^7\theta \cdot d\theta}{\int_{\pi/2}^0 \sin^7\theta \cos^5\theta \cdot d\theta}$$

$\bar{x} =$

$\cos\theta = ct$

$$\bar{x} = \frac{a}{\cancel{a^2}} \frac{\int_0^{\pi/2} \sin^7\theta \cos^8\theta \cdot d\theta}{\int_0^{\pi/2} \sin^7\theta \cos^5\theta \cdot d\theta} = \frac{a}{\cancel{a}} \frac{\dfrac{\cancel{\Gamma 4}\,\Gamma 9/2}{\cancel{2}\,\Gamma \frac{17}{2}}}{\dfrac{\cancel{\Gamma 4}\,\Gamma 3}{\cancel{2}\,\Gamma 7}}$$

$$\bar{x} = \frac{\cancel{8}}{\cancel{2}}\, a \frac{\Gamma 9/2}{\Gamma 17/2} \times \frac{\Gamma 7}{\Gamma 3}$$

$$\bar{x} = a \frac{\lfloor 6}{\lfloor 2} \times \frac{7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \sqrt{\pi}}{15/2 \cdot 13/2 \cdot 11/2 \cdot 9/2 \cdot \cancel{7/2} \cdot \cancel{5/2} \cdot \cancel{3/2} \cdot \cancel{1/2} \cancel{\sqrt{\pi}}}$$

$$\bar{x} = a \frac{\cancel{6} \times \cancel{5} \times 4 \times \cancel{3} \times 2}{\cancel{2}} \times \frac{2 \times 2 \times 2 \times \cancel{2}}{\cancel{15} \times 13 \times 11 \times \cancel{9}\,3}$$

$$\bar{x} = \frac{a\ 16 \times 2 \times 2 \times 2}{429} = \frac{a\,128}{429}$$

$$\boxed{\bar{x} = \frac{128\,a}{429}}$$

$$\bar{y} = \frac{\int_0^a \int_0^y y(kxy) \cdot dx\,dy}{\int_0^a \int_0^y (kxy) \cdot dx\,dy}$$

$$\bar{y} = \frac{\int_0^a (kx) \left[\int_0^y y^2 \cdot dy \right] dx}{\int_0^a (kx) \left[\int_0^y y \cdot dy \right] dx}$$

$$\bar{y} = \frac{\int_0^a \frac{kxy^3}{3} \cdot dx}{\int_0^a \frac{kxy^2}{2} \cdot dx} = \frac{2 \int_0^a xy^3\,dx}{3 \int_0^a xy^2 \cdot dx}$$

$x = a\cos^3\theta$, $y = b\sin^3\theta$

$dx = -3a\cos^2\theta\sin\theta \cdot d\theta$

13
13
143
3
×
429

$$\bar{y} = \left(\frac{2}{3}\right) \frac{\int_{\pi/2}^{0} (a\cos^3\theta)(b^3\sin^9\theta)\cdot(-3a\cos^2\theta\sin\theta)\,d\theta}{\int_{\pi/2}^{0} (a\cos^3\theta)(b^2\sin^6\theta)(-3a\cos^2\theta\sin\theta)\cdot d\theta}$$

$$\bar{y} = \left(\frac{2}{3}\right)\left(\frac{-3a^2b^3}{-3a^2b^2}\right) \frac{\int_{\pi/2}^{0} \cos^5\theta\sin^{10}\theta\cdot d\theta}{\int_{\pi/2}^{0} \cos^5\theta\sin^{7}\theta\cdot d\theta}$$

$$\bar{y} = \left(\frac{2}{3}\right) b \frac{\int_0^{\pi/2} \sin^{10}\theta\cos^5\theta\cdot d\theta}{\int_0^{\pi/2} \sin^7\theta\cos^5\theta\cdot d\theta}$$

$$\bar{y} = \left(\frac{2}{3}\right) b \cdot \frac{\dfrac{\Gamma\frac{11}{2}\,\Gamma 3}{2\,\Gamma 17/2}}{\dfrac{\Gamma 4\,\Gamma 3}{2\,\Gamma 7}}$$

$$\bar{y} = \left(\frac{b}{3}\right) 2 \frac{9/2\cdot 7/2\cdot 5/2\cdot 3/2\cdot 1/2\cdot\sqrt{\pi} \times 6\times 5\times 4\times 3\times 2\times 1}{3\times 2\times 1 \times \frac{15}{2}\times\frac{13}{2}\times\frac{11}{2}\times\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}$$

$$\bar{y} = \left(\frac{b}{3}\right)\frac{64\times 2}{13\times 11} = \frac{128b}{429}$$

Q. Find the total mass of the region in the cube $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$ with density at any point given by xyz.

Q. Find the mass, centroid of tetrahedron bounded by the coordinate plane & the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

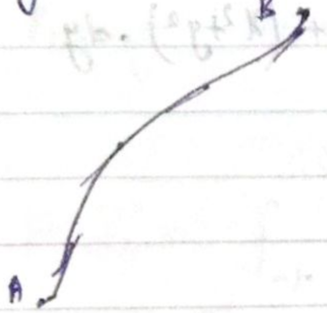
Ans (i) $\rho = xyz$

$$\text{Mass} = \int_0^1\int_0^1\int_0^1 (xyz)\,dx\,dy\,dz = \frac{1}{8}\ \text{Ans}$$

Vector Calculus :—

(i) line Integral ($\int$) (ii) Surface Integral ($\iint$)
(iii) Volume integral ($\iiint$)

(1) Line integral $\Rightarrow$ Any integral which is to be ~~irrot~~ evaluated along the curve is called line integral



Application:—
(i) Total work done
(ii) Conservation field
(iii) Scalar potential

$\int_c \vec{F}\cdot\vec{dr}$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

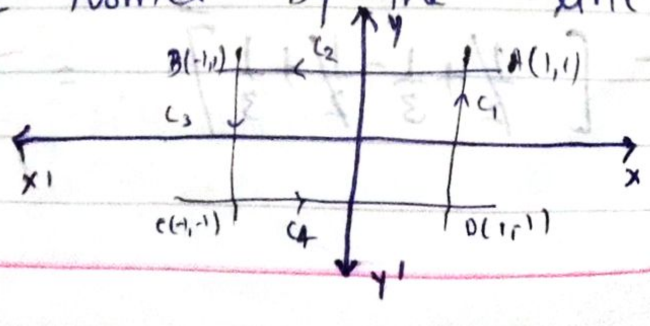
Line integral for closed curve (Contour)

$\oint_c \vec{F}\cdot\vec{dr}$

Q. Evaluate the line integral ~~$\int_c ((x^2$~~

$\int_c (x^2+xy)\,dx + (x^2+y^2)\,dy$, where c is the square formed by the line $x = \pm 1$, $y = \pm 1$

Ans -



$$\int_C \vec{F}\cdot\vec{dr} = \int_{C_1}\vec{F}\cdot\vec{dr} + \int_{C_2}\vec{F}\cdot\vec{dr} + \int_{C_3}\vec{F}\cdot\vec{dr} + \int_{C_4}\vec{F}\cdot\vec{dr}$$

$$\vec{F}\cdot\vec{dr} = \left((x^2+xy)\hat{i} + (x^2+y^2)\hat{j}\right)\left(dx\,\hat{i} + dy\,\hat{j}\right)$$

(i) Along curve C_1

y varies from -1 to 1

& $x = 1 \Rightarrow dx = 0$

$$\int_{C_1}\vec{F}\cdot\vec{dr} = \int_{-1}^{1}(x^2+xy)dx + (x^2+y^2)\cdot dy$$

$$= \int_{-1}^{1}(1+y^2)\cdot dy$$

$$= 2\int_{0}^{1}(1+y^2)\cdot dy = 2\left[y + \frac{y^3}{3}\right]_0^1$$

$$2\left(1+\frac{1}{3}\right) = \frac{8}{3}$$

(ii) Along curve C_2

$$\int_{C_2}\vec{F}\cdot\vec{dr} = \int_{1}^{-1}(x^2+xy)dx + (x^2+y^2)dy$$

x varies 1 to -1

& $y = 1 \Rightarrow dy = 0$

$$= -\int_{-1}^{1}(x^2+x)dx = -\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= -\left[\frac{1}{2}+\frac{1}{3} - \left(\frac{1}{2} - \frac{1}{3}\right)\right]$$

$$= -\left[\cancel{\frac{1}{2}} + \frac{1}{3} - \cancel{\frac{1}{2}} + \frac{1}{3}\right] = -\frac{2}{3}$$

(iii) Along curve C_3

$y=1$ to $y=-1$

& $x=-1$ & $dx=0$

$$\int_{C_3}\vec{F}\cdot\vec{dr} = \int_{1}^{-1}(x^2+xy)\,dx + (x^2+y^2)\,dy$$

$$= -\int_{-1}^{1}(1+y^2)\cdot dy = -2\left[y+\frac{y^3}{3}\right]_0^1$$

$$= -\frac{8}{3}$$

(iv) Along curve C_4

$x=-1$ to $x=1$

$y=-1$ & $dy=0$

$$\int_{C_4}\vec{F}\cdot\vec{dr} = \int_{-1}^{1}(x^2+xy)\,dx + (x^2+y^2)\cdot dy$$

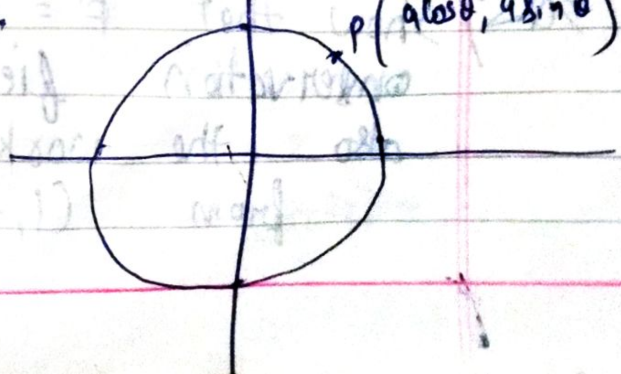
$$\int_{-1}^{1}(x^2-x)\cdot dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{1}$$

$$\frac{1}{3}-\frac{1}{2}-\left(\frac{-1}{3}-\frac{1}{2}\right) = \frac{2}{3}$$

$$\int_C \vec{F}\cdot\vec{dr} = \frac{8}{\cancel{3}} - \frac{2}{\cancel{3}} + \left(-\frac{8}{\cancel{3}}\right) + \frac{2}{\cancel{3}} = 0$$

Total work done of a partical will be zero.

Q. A vector field is given by $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$. Evaluate $\oint \vec{F}\cdot\vec{dr}$ over a circular path c given by $x^2+y^2=a^2$, $z=0$.



Ans. We have

$\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$

$\vec{dr} = dx\,\hat{i} + dy\,\hat{j}$

Page No.:
Date:

$$\oint_c \vec{F}\cdot\vec{dr} = \oint_c [\sin y\,\hat{i} + x(1+\cos y)\hat{j}]\cdot[dx\,\hat{i} + dy\,\hat{j}]$$

$$= \oint_c \sin y\cdot dx + x(1+\cos y)\cdot dy$$

$$= \oint_c \sin y\,dx + x\,dy + x\cos y\,dy$$

$$= \oint_c d(x\sin y) + x\,dy$$

$x = a\cos\theta$, $y = a\sin\theta$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi} d(a\cos\theta\sin(a\sin\theta)) + (a\cos\theta)(a\cos\theta\,d\theta)$$

$$\int_0^{2\pi} d(a^2\cos\theta\sin(\sin\theta)) + \int_0^{2\pi} a^2\cos^2\theta\,.d\theta$$

$$\int_0^{2\pi} d(a^2\cos\theta\sin(\sin\theta)) + \int_0^{2\pi} a^2\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$[a^2\cos\theta\sin(\sin\theta)]_0^{2\pi} + \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$$[a^2\cos\theta\sin(\sin\theta)]_0^{2\pi} + \frac{a^2}{2}[2\pi + 0]$$

$$= \pi a^2 \quad \text{Ans}$$

Q. Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3z^2x\hat{k}$ is a conservation field. Find its scalar potential & also the work done in moving a particle from $(1,-2,1)$ & $(3,1,4)$

POPU

Page No.
Date

Note→ (For a conservative field the work done by a force is independent of path and depends on end points of path.)

Solution $\vec{F}\ (= \nabla\phi)$ is conversation field if
$\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3z^2x \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(3z^2x) - \frac{\partial}{\partial z}(x^2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(3z^2x) - \frac{\partial}{\partial z}(2xy+z^3)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy+z^3)\right)$$

$$= 0\cdot\hat{i} + 0\cdot\hat{j} + 0\cdot\hat{k}$$

so $\vec{F}$ is conservative field

For scalar potential:—
$\vec{F} = \nabla\phi$, $\phi \rightarrow$ Scalar potential

$$(2xy+z^3)\hat{i} + (x^2)\hat{j} + (3z^2x)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing $\hat{i}, \hat{j}$ & $\hat{k}$

$$\frac{\partial\phi}{\partial x} = 2xy+z^3 , \quad \frac{\partial\phi}{\partial y} = x^2 , \quad \frac{\partial\phi}{\partial z} = 3z^2x$$

By total derivative

$$d\phi = \frac{\partial\phi}{\partial x}.dx + \frac{\partial\phi}{\partial y}.dy + \frac{\partial\phi}{\partial z}.dz$$

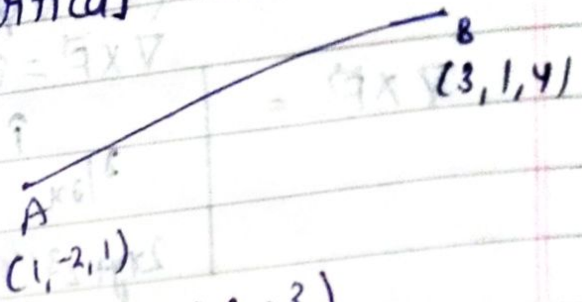
$$d\phi = (2xy+z^3)dx + (x^2)dy + (3z^2x)dz$$
$$\int d\phi = \int (2xy+z^3)dx + (x^2)dy + (3z^2x)dz$$

x^2y+xz^3

$$\int d\phi = \int \cancel{d(x^2yz^3)}\ d(x^2y + xz^3)$$

$$\boxed{\phi = x^2y + xz^3}$$

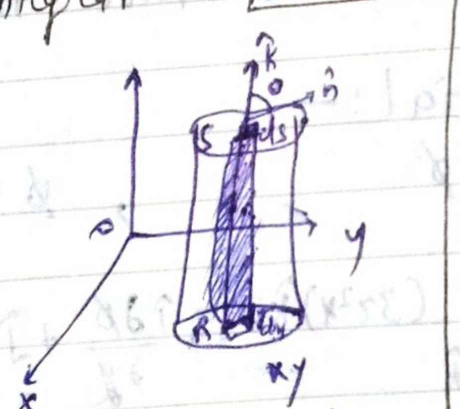
Work done by partical



$$= \phi(B) - \phi(A)$$
$$= \phi(3,1,4) - \phi(1,-2,1)$$
$$= (3)^2(1) + (3)(4)^3 - \left((1)^2(-2) + (1)(1)^3\right)$$
$$= 9 + 192 + 2 - 1$$
$$= 202 \text{ Units}$$

14
×3
112

Surface integral ~~Volume integral~~



eg-24
Q.2
eg-23, 24

Surface integral is given by $= \iint_R \vec{F}\cdot\hat{n}\, ds$ --- (1)

Let R be orthogonal ($\perp$) projection of the surface S on the XY plane.

Now the projection of elementary surface ds on the XY plane

$$= dx\,dy = ds\cos\theta$$

$$ds = \frac{dx\,dy}{\cos\theta} \quad \text{--- (2)}$$

$$\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\hat{n}\cdot\hat{k} = |\hat{n}||\hat{k}|\cos\theta$$

$$\hat{n}\cdot\hat{k} = 1\cdot 1\,\cos\theta$$

$$\cos\theta = |\hat{n}\cdot\hat{k}| \quad \text{--- (3)}$$

from (2)

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad \text{--- (4)}$$

$$\boxed{\text{Surface integral} = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}}$$

Volume integral :—

Volume integral

(i) $\iiint_V \vec{F}\cdot dv = \iiint_V \vec{F}\,dx\,dy\,dz$

(ii) In the component form :—

$$\iiint_V \vec{F}\cdot dv = \hat{i}\iiint_V \vec{F_1}\,dx\,dy\,dz + \hat{j}\iiint \vec{F_2}\,dx\,dy\,dz + \hat{k}\iiint \vec{F_3}\,dx\,dy\,dz$$

$$(\because \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

(iii) $\iiint_V \nabla\times\vec{F}\,dv$

(iv) $\iiint_V \phi\,dv$

POPU

Page No. :
Date : / /

✱ Green's theorem :—

If $M(x,y)$, $N(x,y)$ are continuous functions of x & y having continuous first order partial derivative in a Region R of xy -Plane bounded by a closed curve c, then

$$\oint_c (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

✱ Stoke's theorem :—

If S be an open surface bounded by a closed curve c and $\vec{F}$ be any vector point function having continuous, first order partial derivatives

$$\oint_c \vec{F}\cdot\vec{dr} = \iint_s \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$ [Relation b/w line and surface integral]

✱ Gauss's theorem :—

Let S be a closed surface ~~integral~~ enclosing a region of space with volume V & F be any vector point function having continuous, first order partial derivatives in V, then

$$\iint_s \vec{F}\cdot\hat{n}\,ds = \iiint_v (\text{div}\,\vec{F})\cdot dV$$

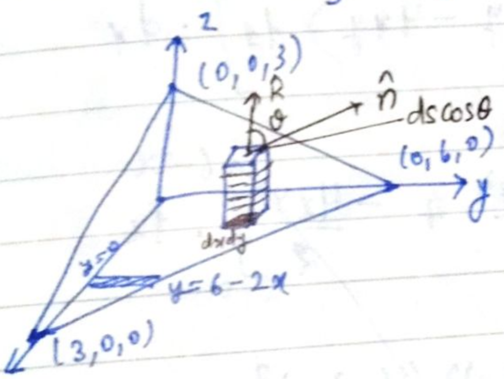
Q. Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$, where $\vec{F} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and

S is surface of the plane $2x+y+2z=6$ in the first octant.

Ans- plane $2x + y + 2z = 6$

$$\frac{2x}{6} + \frac{y}{6} + \frac{2z}{6} = 1$$

$$\frac{x}{3} + \frac{y}{6} + \frac{z}{3} = 1 \qquad \left(\because \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1\right)$$



$$ds\cos\theta = dx\,dy$$

$$ds = \frac{dx\,dy}{\cos\theta}$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\vec{n} = \text{grad } S = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+y+2z-6)$$

$$\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}}$$

$$\hat{n} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$\hat{n}\cdot\hat{k} = \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)\cdot\hat{k} = \frac{2}{3}$$

$$|\hat{n}\cdot\hat{k}| = \frac{2}{3}$$

Now $\iint_S \vec{F}\cdot\hat{n}\, ds$

$$= \iint \left[(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\right]\left(\frac{2\hat{i}}{3} + \frac{\hat{j}}{3} + \frac{2\hat{k}}{3}\right)\left(\frac{dx\,dy}{2/3}\right)$$

POPU

Page No.:
Date:

$$= \frac{1}{2}\iint \left[2(x+y^2) - 2x + 4yz \right] dx\,dy$$

$$\left\{ \because z = \tfrac{1}{2}(6-2x-y) \right\}$$

$$= \frac{1}{2}\iint \left[\cancel{2x} + 2y^2 - \cancel{2x} + 4y\left(\tfrac{1}{2}(6-2x-y)\right)\right] dx\,dy$$

$$= \frac{1}{2}\iint \left[\cancel{2x} \cdot \left(2y^2 + 12y - 4xy - 2y^2\right) dy\,dx\right.$$

$$= \frac{1}{2}\int_0^3 \left[\int_0^{6-2x} \left(\cancel{-2y^3} + 12y - 4xy \right) dy \right] \cdot dx$$

$$= \frac{1}{2}\int_0^3 \left[\frac{-2y^3}{3} + \frac{12y^2}{2} + \frac{-4xy^2}{2} \right]_0^{6-2x} \cdot dx$$

$$= \frac{1}{2}\int_0^3 \left[\frac{-2(6-2x)^3}{3} + \frac{12(6-2x)^2}{2} - 2x(6-2x)^2 \right] \cdot dx$$

$$= \frac{1}{2}\int_0^3 -\frac{16}{3}(3-x)^3 + 24(3-x)^2 - 8x(9-6x+x^2) \Big] \cdot dx$$

$$= \frac{1}{2}\int_0^3 \left(24(9-6x+x^2) - 72x + 48x^2 + 8x^3 \right) dx$$

$$= \frac{1}{2}\int_0^3 \left(216 - 144x + 72x^2 - 72x + 8x^3 \right) \cdot dx$$

$$= \frac{1}{2}\int_0^3 \left(72x^2 - 216x + 216 \right) \cdot dx$$

$$= \frac{72}{2}\int_0^3 \left(x^2 - 3x + 1 \right) dx$$

$$= 36 \left[\frac{x^3}{3} - \frac{3x^2}{2} + x \right]_0^3$$

$$= 36\left[\frac{27}{\cancel{3}}^{9} - \frac{27}{2} + 3\right]$$

$$= \cancel{36}^{18}\left[\frac{54-27}{\cancel{2}}\right]$$

$$\frac{1}{2}\int_0^3 \int_0^{6-2x} 4y(3-x)\,.\,dx\,dy = 2\int_0^3 (3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} .\,dx$$

$$= \frac{2}{2}\int_0^3 (3-x)\,(6-2x)^2\,.\,dx$$

$$= \cancel{4}\int_0^3 (3-x)3\,.\,dx = 2\int_0^3 27 - \cancel{27x} + \cancel{9x^2} \ldots + x^3\,.\,dx$$

$$\cancel{4}\cancel{\left[\frac{x^3}{3} - 3x^2 + 9x\right]_0^3} = \cancel{\frac{927}{3} - 27 + 27}$$

$$\left[27x - \frac{27x^2}{2} + 3x^3 + \frac{x^4}{4}\right]_0^3 = 81$$

Q. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ then.
evaluate (i) $\iiint_v \nabla \times \vec{F}\,.\,dv$

(ii) $\iiint_v \nabla\cdot\vec{F}\,.\,dv$ where v is closed region bounded by the planes $x=0, y=0, z=0$ and $2x + 2y + z = 4$

Ans - $\iiint_v \nabla\times\vec{F}\,.\,dv =$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$\nabla\times\vec{F} = \hat{i}\left(\partial/\partial y(-4x) - \frac{\partial}{\partial z}(-2xy)\right) - \hat{j}\left[\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2x^2 - 3z)\right)$$

$$+\hat{k}\left(\partial/\partial x(-2xy) - \partial/\partial y(2x^2 - 3z)\right)$$

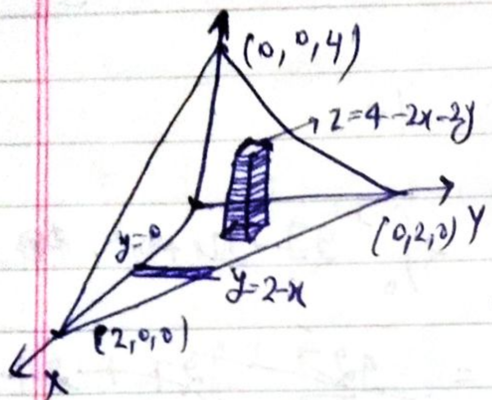
$$= +4\hat{j} - 3\hat{j} - 2y\hat{k} = 1\hat{j} - 2y\hat{k}$$

plane

$$2x + 2y + z = 4.$$

$$\frac{2x}{4} + \frac{2y}{4} + \frac{z}{4} = 1$$

$$\frac{x}{2} + \frac{y}{2} + \frac{z}{4} = 1 \quad \left\{ \because \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \right\}$$



(i) $\iiint \nabla \times \vec{F}\, dV$

$$= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (\hat{j} - 2y\hat{k})\, dx\, dy\, dz$$

$$= + \hat{j} \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} dx\,dy\,dz - 2\hat{k} \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y\, dx\,dy\,dz$$

$$= \hat{j} \int_0^2 \int_0^{2-x} \left[\int_0^{4-2x-2y} dz \right] dx\,dy - 2\hat{k} \int_0^2 \int_0^{2-x} y \left[\int_0^{4-2x-2y} dz \right] dx\,dy$$

$$= \hat{j} \int_0^2 \left[\int_0^{2-x} 4 - 2x - 2y\, dy \right] dx - 2\hat{k} \int_0^2 \left[\int_0^{2-x} 4y - 2xy - 2y^2 \cdot dy \right] dx$$

$$= \hat{j} \int_0^2 \left[4y - 2xy - y^2 \right]_0^{2-x} dx - 2\hat{k} \int_0^2 \left(2y^2 - xy^2 - \frac{2y^3}{3} \right)_0^{2-x} dx$$

$$= \hat{j} \int_0^2 4(2-x) - 2x(2-x) - (2-x)^2 \cdot dx - 2\hat{k} \int_0^2 2(2-x)^2 - x(2-x)^2 - \frac{2(2-x)^3}{3} dx$$

$$= \hat{j} \int_0^2 (\underbrace{8 - 8x + 2x^2 - 4 + 4x - x^2}_{4 - 4x + x^2}) \cdot dx - 2\hat{k} \int_0^2 2(4 - 4x + x^2) - x(4 - 4x + x^2) - \frac{2}{3}(8 - 12x + 6x^2 - x^3) \cdot dx$$

$$= \hat{j}\left[\cancel{16}-\cancel{16}-\cancel{\frac{2}{3}(2)^2}-\cancel{8}+\cancel{8}+\frac{8}{3}\right]-2\hat{k}\left[16-4(2)^2+\frac{16}{3}-8+\frac{4}{3}(2)^3-\frac{(2)^4}{4}-\frac{2}{3}\left(16-16+\frac{4}{3}\times 8-\frac{(2)^4}{4}\right)\right]$$

$$\frac{8}{3}\hat{j}$$

$$= \hat{j}\left[\cancel{4-\frac{16}{3}}\right]-2\hat{k}\left[\frac{16}{3}-8+\frac{32}{3}-\frac{16}{4}-\frac{32}{3}+\frac{32}{3}-\frac{64}{9}+\frac{32}{3\times 4}\right]$$

$$= \cancel{-\frac{4}{3}\hat{j}}-2\hat{k}\left[\frac{64}{3}-\frac{64/9}{}-12+\frac{8}{3}\right] \quad \frac{64}{3}-\frac{64}{9}-12$$

$$\frac{8}{3}\hat{j}-2\hat{k}\left[\cancel{\frac{-16+56}{3}}\right] = -\frac{56}{8}\times 2\;\cancel{\frac{182-64-108}{8}}$$

$$\cancel{\frac{8}{3}\hat{j}\;0+\frac{40}{3}\hat{k}\times 2}$$

$$\cancel{\frac{8}{3}\hat{j}-2\hat{k}\left(-\frac{4}{9}\right)} = \frac{8}{3}\hat{j}+\frac{8\hat{k}}{9}$$

$$\frac{8}{3}\hat{j}-\frac{8}{3}\hat{k} = \frac{8}{3}(\hat{j}-\hat{k})$$

(ii) $\iiint \nabla\cdot\vec{F}\,dV$

$$= \int_0^2\int_0^{2-x}\int_0^{4-2x-2y}\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left[(2x^2-3z)\hat{i}-2xy\hat{j}-4x\hat{k}\right]dV$$

$$= \int_0^2\int_0^{2-x}\int_0^{4-2x-2y}(4x-2x)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^{2-x}(2x)\,dy\,dx\left[4-2x-2y\right]$$

$$4\int_0^2 x\,dx\left[\int_0^{2-x}(2-x-y)\,dy\right]$$

$$4\int_0^2 x\,dx \left[2y - xy - \frac{y^2}{2}\right]_0^{2-x}$$

$$4\int_0^2 x\,dx \left[2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\right]$$

$$4\int_0^2 x\left(4 - 2x - 2x + x^2 - \frac{4+4x-x^2}{2}\right).dx$$

$$\frac{4}{2}\int_0^2 x\left(8 - 8x + 2x^2 - 4 + 4x - x^2\right).dx$$

$$= 2\int_0^2 x\left(4 - 4x + x^2\right).dx = \frac{1}{2}\int_0^2 4x - 4x^2 + x^3 .dx$$

$$\cancel{4 \times \frac{1}{2}\left[x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right]_0^2}$$

$$\cancel{= \frac{1}{2}\left[4 - \frac{32}{3} + \frac{16}{4}\right] = \frac{24-32}{3} \times \frac{1}{2}}$$

$$\cancel{= \frac{-20}{3}}$$

$$2\left[\frac{4x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4}\right]_0^2$$

$\frac{-4}{3}$

$$8 - \frac{32}{3} + 4 = 12 - \frac{32}{3} = \frac{4}{3} \times 2$$

$$= \frac{8}{3}$$

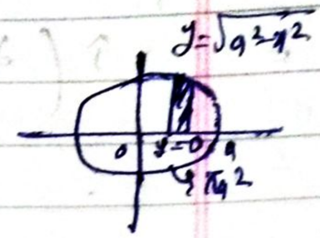
Q. Use Green's theorem in a plane to evaluate $\oint_c (2x-y)dx + (x+y)dy$ where C is the boundary of circle $x^2+y^2=a^2$ in the XY plane.

Ans- By Green's theorem

$$\oint P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$P = 2x - y \quad , \quad Q = x + y$

$\frac{\partial P}{\partial y} = -1 \quad , \quad \frac{\partial Q}{\partial x} = 1$



$$\oint_c [2x-y]\,dx + (x+y)\,dy = \iint_R [1-(-1)]\,dx\,dy$$

$= 2\iint_R dx\,dy$

$= 2 \times$ Area of circle in XY plane

$= 2 \times \pi a^2$

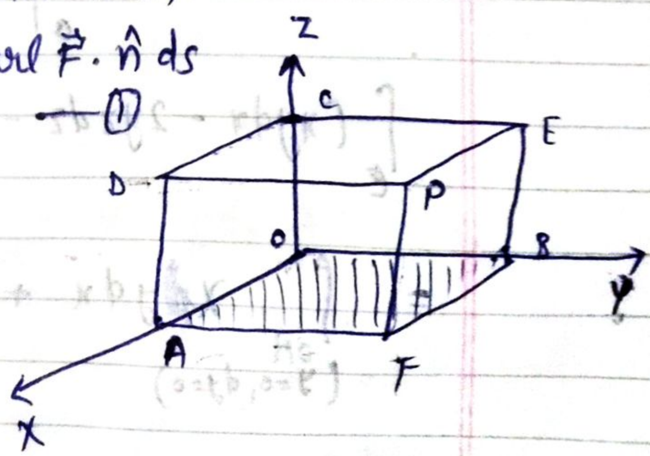
$= 2\pi a^2$

Q. Verify stoke's theorem for $\vec{F} = xy\hat{i} - 2yz\hat{j} - zx\hat{k}$ where S is the open surface of rectangle parallelopiped formed by the planes $x=0$, $x=1$, $y=0$, $y=2$ & $z=3$ above the XY plane.

Ans-

Stoke's theorem state that

$$\int_c \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds \quad \text{--- (1)}$$



$\vec{F} = xy\hat{i} - 2yz\hat{j} - zx\hat{k}$

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -2yz & -zx \end{vmatrix}$$

$$= \hat{i}\left(\partial/\partial y(-zx) - \frac{\partial}{\partial z}(-2yz)\right) - \hat{j}\left(\frac{\partial}{\partial x}(-zx) - \frac{\partial}{\partial z}(xy)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(-2yz) - \frac{\partial}{\partial y}(xy)\right)$$

$$= 2y\,\hat{i} - \hat{j}\,(-z) + \hat{k}\,(-x)$$

$$2y\,\hat{i} + z\,\hat{j} - x\,\hat{k}$$

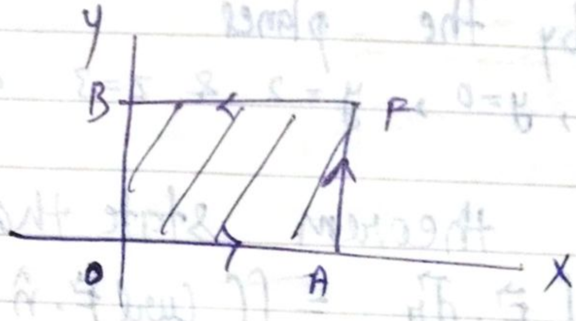
$$\vec{F}\cdot d\vec{r} = (xy\,\hat{i} - 2yz\,\hat{j} - zx\,\hat{k})\,(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= xy\,dx - 2yz\,dy - zx\,dz$$

from 1

$$\int_C (xy\,dx - 2yz\,dz - zx\,dz) = \iint (2y\,\hat{i} + z\,\hat{j} - x\,\hat{k})\cdot\hat{n}\,ds \quad (2)$$

LHS:—



$$\int_C (xy\,dx - 2yz\,dz - zx\,dz) = \int_C (xy\,dx)$$

$$= \int_{\substack{OA \\ (y=0,\ dy=0)}} xy\,dx + \int_{\substack{AF \\ (x=1,\ dx=0)}} xy\,dx + \int_{\substack{PB \\ (y=2,\ dy=0)}} xy\,dx + \int_{\substack{OB \\ (x=0,\ dx=0)}} xy\,dx$$

$$= 0 + 0 \int_{FB} 2x\,dx + 0 = 2\int_1^0 x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_1^0 = -2\times\frac{1}{2} = -1$$

RHS :— $\iint_R (2y\hat{i} + z\hat{j} - x\hat{k})\hat{n}\,ds$

$$= \iint_{\substack{OBEC \\ (x=0,\ \hat{n}=-\hat{i})}} (2y\hat{i}+z\hat{j}-x\hat{k})\hat{n}\,ds + \iint_{\substack{(AFPD) \\ (x=1,\ \hat{n}=\hat{i})}} (2y\hat{i}+z\hat{j}-x\hat{k})\hat{n}\,ds$$

$$+ \iint_{\substack{(AOCD) \\ (y=0,\ \hat{n}=-\hat{j})}} (2y\hat{i}+z\hat{j}-x\hat{k})\hat{n}\,ds + \iint_{\substack{(y=2)\ (PEBF) \\ (\hat{n}=\hat{j})}} (2y\hat{i}+z\hat{j}-x\hat{k})\hat{n}\,ds$$

$$+ \iint_{\substack{PDCE \\ (z=3,\ \hat{n}=\hat{k})}} (2y\hat{i}+z\hat{j}-x\hat{k})\hat{n}\,ds$$

$$= \iint (2y\hat{i}+z\hat{j})(-\hat{i})\,ds + \iint (2y\hat{i}+z\hat{j}-x\hat{k})\,\hat{i}\,ds$$

$$+ \iint (z\hat{j}-x\hat{k})(-\hat{j})\,ds + \iint (4\hat{i}+z\hat{j}-x\hat{k})(\hat{j})\,ds$$

$$+ \iint (2y\hat{i}+3\hat{j}-x\hat{k})\,\hat{k}\,ds$$

$$= \iint -2y\,\cancel{ds} + \iint \cancel{2y}\,ds - \int \cancel{z}ds + z\iint \cancel{ds}$$

$$- \iint x\,ds$$

$$= -\iint x\,dx\,dy = -\int_0^1 x\int_0^2 dy\cdot dx$$

$$-\int_0^1 [2x]\,dx = -\left[x^2\right]_0^1 = -1$$

Q. explain by Green's theorem:—

$$\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,1)$, $(0,1)$

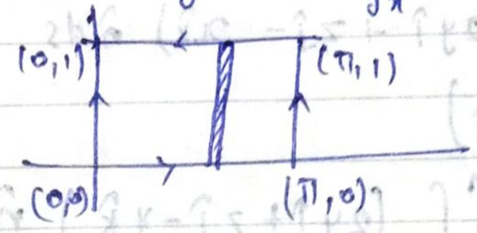
Ans- Green theorem

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = x^2 - \cosh y$

$N = y + \sin x$

$$\frac{\partial M}{\partial y} = -\sinh y \quad , \quad \frac{\partial N}{\partial x} = \cos x$$



$$\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$$

$$= \int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x + \sinh y)\,dx\,dy$$

$$= \int_{x=0}^{\pi}\left[\int_{y=0}^{1} (\cos x + \sinh y)\,dy\right] dx$$

$$= \int_{x=0}^{\pi} \left[y\cos x + \cosh y\right]_0^1 dx$$

$$= \int_{x=0}^{\pi} \cos x + \cosh - (0 + \cos 0)\Big]\,dx$$

$$\int_{x=0}^{\pi} (\cos x + \cosh - 1)\,dx$$

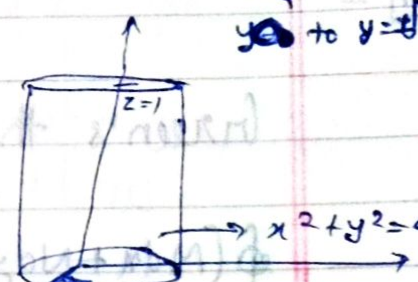
$$\left[\sin x + x\cosh x\right]_0^{\pi}$$

$$0 + \pi\cosh - \pi = \pi(\cosh - 1) \quad \text{Ans}$$

Q. If $\vec{F} = x\hat{i} - y\hat{j} + (z^2-1)\hat{k}$ then find the value of $\iint_S \vec{F}\cdot\hat{n}\,ds$ where S is closed surface bounded by $z=0$, $z=1$ and the cylinder $x^2+y^2=4$

$x=0$ to $x=2$

$y=0$ to $y=\pm\sqrt{4-x^2}$



Ans – By Gauss' divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x\hat{i} - y\hat{j} + (z^2-1)\hat{k}\right) dv$$

$$= \iiint_V \frac{\partial x}{\partial x} + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2-1)\ \cdot dv$$

$$= \iiint (1 - 1 + 2z)\ dx\,dy\,dz$$

$$= \iiint 2z\ dx\,dy\,dz$$

$$= \int_{x=0}^{\pm 2}\int_{y=0}^{\pm\sqrt{4-x^2}}\int_{z=0}^{1} 2z\ dx\,dy\,dz$$

$$= \int_{x=0}^{\pm 2}\int_{y=0}\left[\int_{y=0}^{\pm\sqrt{4-x^2}}\int_{z=0}^{1} 2z\,dz\right] dx\,dy = 2\int_{x=0}^{2}\sqrt{4-x^2}\cdot dx$$

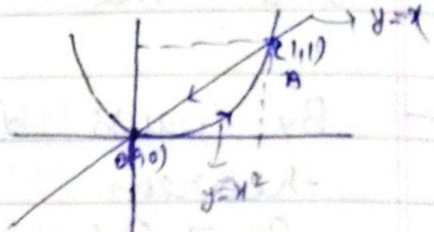
$$2\frac{z^2}{2} \qquad [y]_0^{\sqrt{4-x^2}} = \sqrt{4-x^2}$$

$$4\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$4 \times \cancel{2} \times \frac{\pi}{\cancel{2}} = 4\pi \quad \underline{Ans}$$

Q. Verify the Green theorem in the plane $\oint_c (xy+y^2)\,dx + x^2dy$, where c is the closed curve of the region bounded by $y=x$ & $y=x^2$.

Ans –



Green's theorem in plane

$$\oint_c (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{--- (1)}$$

we have $y = x$ — (2)

$y = x^2$ — (3)

$\Rightarrow$

$x^2 = x$

$x(x-1) = 0$

$x = 0, \; x = 1$

$y = 0, \; y = 1$

Intersection pt of curve $(0,0)$ & $(1,1)$

$M = xy + y^2, \quad N = x^2$

$\frac{\partial M}{\partial y} = x + 2y, \quad \frac{\partial N}{\partial x} = 2x$

For LHS part

$$\oint_c (M\,dx + N\,dy) = \int_{OA+AO} (xy+y^2)\,dx + x^2dy$$

POPU

Page No.:

Date:

Along OA :—

$$y = x^2 , \quad dy = 2x \cdot dx$$

$$x = 0 \text{ to } x = 1$$

$$\int_0^1 (x \times x^2 + (x^2)^2) dx + x^2(2x) \cdot dx$$

$$= \int_0^1 (x^3 + x^4 + 2x^3) \cdot dx = \int_0^1 (3x^3 + x^4) dx$$

$$= \left[\frac{3}{4}x^4 + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{15+4}{20}$$

$$= \frac{19}{20}$$

Along AO :—

$$y = x , \quad dy = dx$$

$$x = 1 \text{ to } x = 0$$

$$\int_1^0 (x^2 + x^2) dx + x^2 \cdot dx = \int_1^0 3x^2 \cdot dx$$

$$= \left[\frac{3x^3}{3}\right]_1^0 = -1$$

$$\int (xy + y^2) \cdot dx + x^2 dy = \frac{19}{20} - 1 = \frac{19-20}{20} = \frac{-1}{20} \quad (4)$$

Now for RHS part

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$\int_{x=0}^1 \int_{y=x^2}^{y=x} (2x - x - 2y) \, dx\, dy = \int_{x=0}^1 \left[\int_{y=x^2}^{y=x} (x-2y)\, dy\right] dx$$

$$= \int_{x=0}^1 [xy - y^2]_{y=x^2}^{y=x} \cdot dx$$

$$= \int_{x=0}^1 \cancel{x^2} - \cancel{x^2} - x^3 + x^4 \cdot dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = \frac{-1}{20} \quad (5)$$

from eq^n – ④ & ⑤
the Green's theorem verify.

Q. Apply stoke's theorem to evaluate $\oint_c \vec{F}\cdot\vec{dr}$, where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$

and c is the boundary of the triangle with verties $(0,0,0)$, $(1,0,0)$ $(1,1,0)$.

Ans– By stoke's theorem

$$\oint_c \vec{F}\cdot\vec{dr} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds \quad \text{—①}$$

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

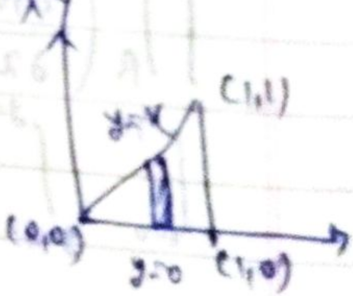
$$= \hat{i}\left(\partial/\partial y(-x-z) - \partial/\partial z(x^2)\right) - \hat{j}\left(\partial/\partial x(-x-z) - \partial/\partial z(y^2)\right)$$
$$+ \hat{k}\left(\partial/\partial x(x^2) - \partial/\partial y(y^2)\right)$$

$$\text{Curl}\,\vec{F} = \hat{j} + (2x-2y)\hat{k}$$

Since triangle lies in xy plane then $\hat{n} = \hat{k}$

$x=0$ to $x=1$

$y=0$ to $y=x$



$$\oint_c \vec{F}\cdot\vec{dr} = \int_{x=0}^{x=1}\int_{y=0}^{y=x} \left[\hat{j} + (2x-2y)\hat{k}\right]\cdot\hat{k}\cdot ds$$

POPU

Page No.:
Date: / /

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=x} (2x-2y)\, dy \right] dx$$

$$\int_{x=0}^{1} \left[2xy - y^2\right]_{y=0}^{y=x} \cdot dx = \int_{x=0}^{1} (2x^2 - x^2) \cdot dx$$

$$\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \text{ u}$$

Q. Evaluate $\oint_C (e^x dx + 2y dy - dz)$; by stoke's theorem, where c is the curve $x^2+y^2=4, z=2$

Ans—

$$\oint_C [e^x dx + 2y dy - dz]$$

$$\vec{F} = e^x \hat{i} + 2y\hat{j} - \hat{k}$$

$$\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds \quad \text{——} \textcircled{1}$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & 2y & -1 \end{vmatrix}$$

$$= 0$$

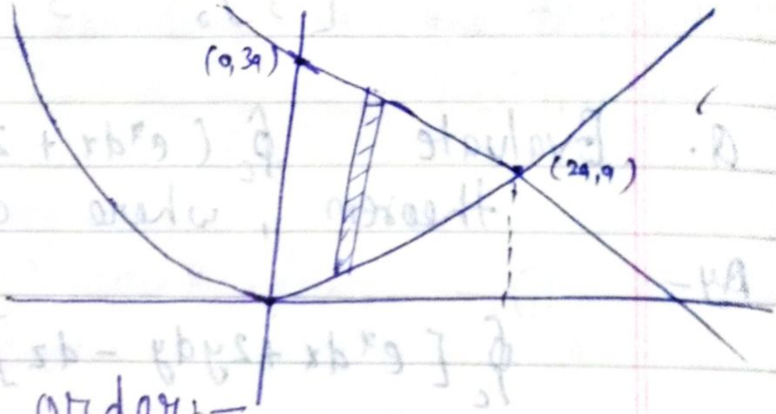
$$\oint_C \vec{F} \cdot \vec{dr} = \iint \text{curl}\,\vec{F} \cdot \hat{n}\, ds = 0 \quad (\because \text{curl}\,\vec{F} = 0)$$

Q. $\int_0^{2a}\int_{x^2/4a}^{3a-x} (x^2+y^2)\,dx\,dy$

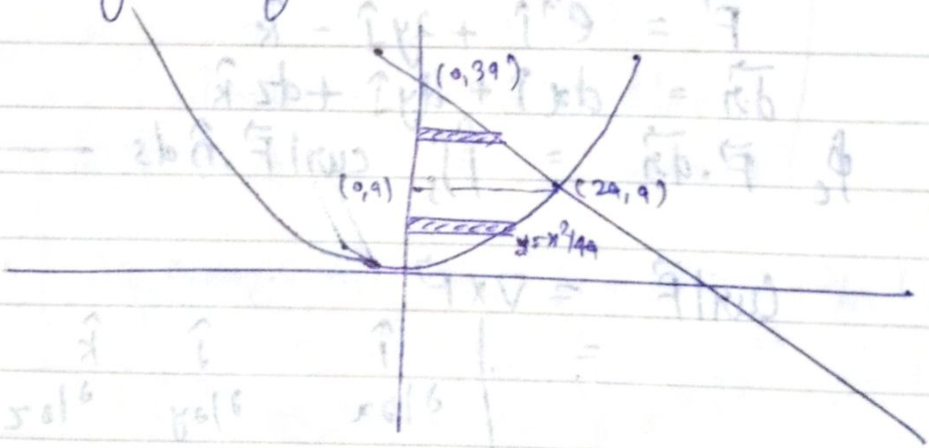
Ans- where $x=0$ to $x=2a$

$y=\frac{x^2}{4a}$ to $y=3a-x$ (Strip ∥ to y-axis)

$x^2=4ay$ to $y=3a-x$



change of order:-



$$\iint_{R_1} f(x,y)\,dy\,dx + \iint_{R_2} f(x,y)\,dy\,dx$$

$$= \int_0^a \int_{x=0}^{2\sqrt{ay}} (x^2+y^2)\,dy\,dx + \int_a^{3a}\int_{x=0}^{3a-y} (x^2+y^2)\,dy\,dx$$

$$\int_0^a dy\left[\int_{x=0}^{2\sqrt{ay}} (x^2+y^2)\,dx\right] + \int_a^{3a}\left[\int_{x=0}^{3a-y} (x^2+y^2)\,dx\right] dy$$

$$\int_0^a dy \left[\frac{x^3}{3} + y^2x\right]_0^{2\sqrt{ay}} + \int_a^{3a} \left[\frac{x^3}{3} + y^2x\right]_0^{3a-y} dy$$

—— x ——

Q. Verify Gauss Divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ bounded by cylinder $x^2+y^2=4$, $z=0$ to $z=3$

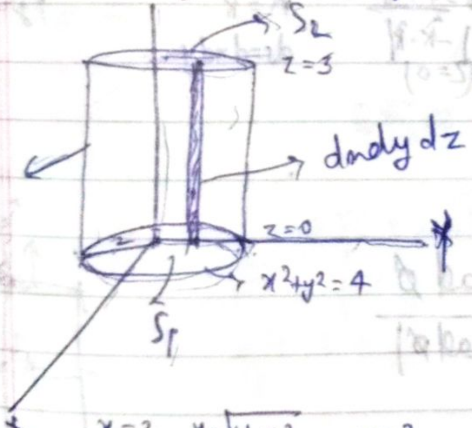
Sol-

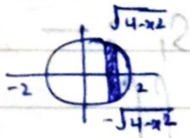
$$\iint_S \vec{F}\cdot\hat{n}\,d\vec{s} = \iiint_V (\mathrm{div}\vec{F})\,dv$$

Here V is the closed region bounded the surface S.

RHS :— $\iiint_V (\mathrm{div}\vec{F})\,dv$

$\mathrm{div}\vec{F} = \nabla\cdot\vec{F}$
$= 4 - 4y + 2z$





where $z=0$ to $z=3$
$y=-\sqrt{4-x^2}$ to $y=\sqrt{4-x^2}$
$x=-2$ to $x=2$

$$\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=0}^{z=3} (4-4y+2z)\,dx\,dy\,dz$$

$$\int_{x=-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4z - 4yz + z^2\right)_0^3\,dx\,dy$$

$$\int_{x=-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\,dy\right]dx$$

$$\int_{x=-2}^{2} 42\sqrt{4-x^2}\; dx$$

$$2\times 42\int_0^2 \sqrt{4-x^2}\; dx$$

$$84\left[\frac{x^2}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$84\times 2\times\frac{\pi}{2} = 84\pi$$

LHS $\Rightarrow$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

S_1: $\hat{n}=-\hat{k}$, $ds=\dfrac{dx\,dy}{|-\hat{k}\cdot\hat{k}|}$ $(z=0)$

S_2: $(z=3)$, $\hat{n}=\hat{k}$, $ds=dx\,dy$

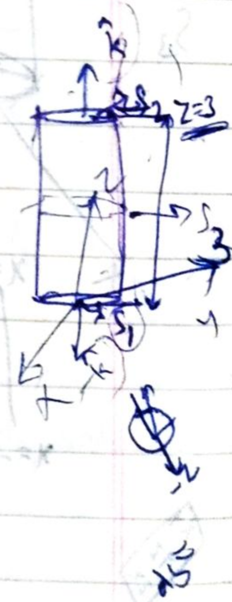
S_3: $\hat{n}=\dfrac{grad\,\phi}{|grad\,\phi|}$

For S_1:

$\iint$

For S_3:

$$\hat{n}=\frac{grad\,\phi}{|grad\,\phi|}$$



$ds = 2\, d\theta\, dz$

$x = 2\cos\theta$

$y = 2\sin\theta$

$\theta = 0$ to $\theta = 2\pi$

$z = 0$ to $z = 3$

$$\int_{\theta=0}^{2\pi}\int_{z=0}^{3} \vec{F}\cdot\hat{n}\,(2\, d\theta\, dz)$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} -z^2\,dx\,dy = 0 \quad \text{(which is in xy plane ie } z=0\text{)} \quad —①$$

$$\iint_{S_2} z^2\,dx\,dy = \iint 9\,dx\,dy = 9\times\pi\times 4 = 36\pi \quad —②$$

(which is ∥ to xy plane at $z=3$)

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds =$$

$$\phi = x^2+y^2-4=0$$

$$\text{grad}\,\phi = 2x\hat{i}+2y\hat{j}$$

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y\hat{j}}{\sqrt{x^2+y^2}}$$

$$= \frac{x\hat{i}+y\hat{j}}{2}$$

$$\int_{\theta=0}^{2\pi}\int_{z=0}^{z=3}\left(4x\hat{i}-2y^2\hat{j}+z^2\hat{k}\right)\left(\frac{x\hat{i}+y\hat{j}}{2}\right)(2\,d\theta\,dz)$$

$$\int_{\theta=0}^{2\pi}\int_{z=0}^{3}\left(4x^2-2y^3\right)d\theta\,dz$$

$$x = 2\cos\theta,\quad y = 2\sin\theta$$

$$\int_{\theta=0}^{2\pi}\left(4\times(2\cos\theta)^2 - 2(2\sin\theta)^3\right)d\theta\left[\int_{z=0}^{3}dz\right]$$

$$3\int_{\theta=0}^{2\pi} 16\left(\cos^2\theta-\sin^3\theta\right)d\theta = 48\int_{\theta=0}^{2\pi}(\cos^2\theta-\sin^3\theta)\,d\theta$$

$$48\int_0^{2\pi} \cos^2\theta \cdot d\theta - 48\int_0^{2\pi} \sin^3\theta \cdot d\theta$$

$$48\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)\cdot d\theta - 48(0)$$

$$\frac{48}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$$24 \times 2\pi = 48\pi \quad \text{— (3)}$$

Add eqn (1) & (2) & (3)

$$0 + 36\pi + 48\pi = 84\pi$$

LHS = RHS

Q. Verify Green's theorem

$$\oint_c (x^2-2xy)\,dx + (x^2y+3)\,dy$$

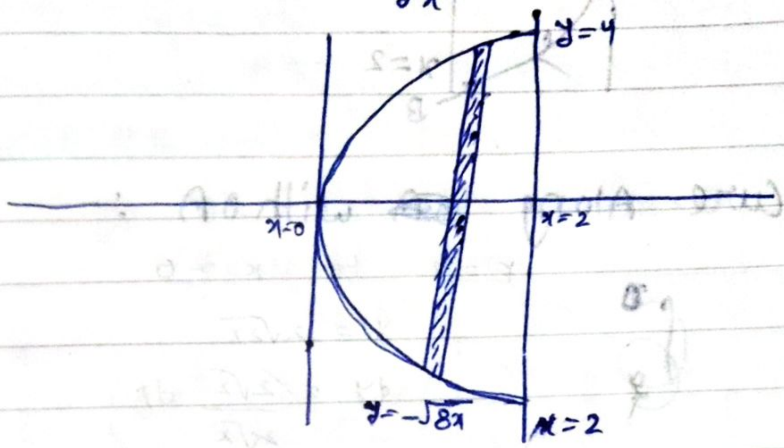
$y^2 = 8x$ and $x = 2$

Ans- Green's theorem

$$\oint_c (M\,dx + N\,dy) = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

$$M = x^2 - 2xy \quad , \quad N = x^2y + 3$$

$$\frac{\partial M}{\partial y} = -2x \quad , \quad \frac{\partial N}{\partial x} = 2xy$$



$$y^2 = 8x = y^2 = 16 \Rightarrow y = 4$$

RHS :—

$$\int_0^2 \left[\int_{-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x)\, dy \right] dx$$

$$\int_0^2 \left[xy^2 + 2xy \right]_{-\sqrt{8x}}^{\sqrt{8x}} dx$$

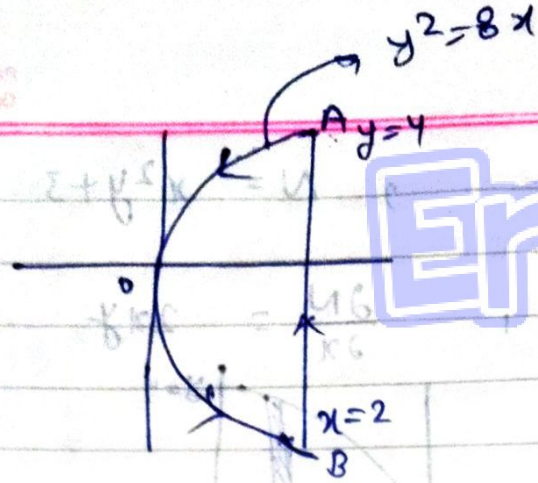
$$\int_0^2 \left\{ x(8x) + 2x\sqrt{8x} - x(-\sqrt{8x})^2 - 2x(-\sqrt{8x}) \right] dx$$

$$\int_0^2 4x\sqrt{8x}\, dx = \int_0^2 8\sqrt{2}\, x^{3/2} \cdot dx$$

$$\frac{8\sqrt{2} \times 2}{5} x^{5/2} = \frac{16\sqrt{2}}{5} (2)^{5/2} \cdot dx = \frac{16 \times 4 \times 2}{5}$$

$$= \frac{128}{5}$$

LHS :—



cure Along with OA :—

$x = 2$ to $x = 0$

$y = 2\sqrt{2x}$

$dy = \frac{2\sqrt{2}}{2\sqrt{x}}\, dx$

$dy = \sqrt{\frac{2}{x}}\, dx$

$$\int_2^0 (x^2 + 4x\sqrt{2x})\,dx + (2x^2\sqrt{2x} + 3)\sqrt{\frac{2}{x}}\,dx$$

$$\int_2^0 \left(x^2 + 4\sqrt{2}\,x^{3/2} + 4x^2 + 3\sqrt{2}\,x^{-1/2}\right)dx$$

$$\int_2^0 \left(3\sqrt{2}\,x^{-1/2} + 4\sqrt{2}\,x^{3/2} + 5x^2\right)dx$$

$$\left[6\sqrt{2}\,x^{1/2} + \frac{8\sqrt{2}}{5}x^{5/2} + \frac{5}{3}x^3\right]_2^0$$

$$-\left[6\sqrt{2}\sqrt{2} + \frac{8\sqrt{2}}{5}\,2\sqrt{2} + \frac{5}{3}\cdot 8\right]$$

$$-\left[12 + \frac{32}{5} + \frac{40}{3}\right]$$

$$-\left[\frac{28}{5} + \frac{40}{3}\right]$$

$$-\frac{284}{15}$$

Q. $\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$ prove

let $c^x = e^t$

$x\log c = t\log_e e$

$t = x\log c$

$dt = \log c \cdot dx$

$x = \frac{t}{\log c}$

$$\int_0^\infty \frac{\left(\frac{t}{\log c}\right)^c}{e^t}\cdot\frac{dt}{\log c}$$

$$\int_0^\infty \frac{t^c}{e^t}\cdot\frac{dt}{(\log c)^{c+1}}$$

$$\frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-t}\cdot t^c\cdot dt$$

$$\frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-t}\cdot t^{(c+1)-1}\cdot dt$$

$$\frac{\Gamma(c+1)}{(\log c)^{c+1}}$$ ans